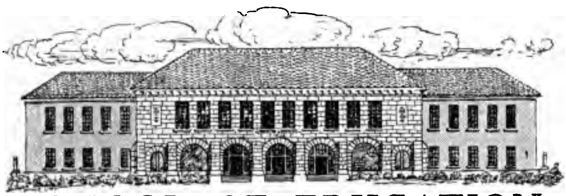

SECONDARY ALGEBRA

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SECONDARY ALGEBRA

BY

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PREFACE.

In the preparation of this book, the aim of the authors has been to give the student a working knowledge of the elementary processes of algebra, with a conviction of the truth of principles through illustrations and particular examples. Each principle, or method, is therefore first clearly illustrated by numerous and simple exercises worked in the text. But the student is not left to assume that the principles are thereby proved. Even a beginner should not be encouraged, by text-book or teacher, to accept an illustrative example as a proof, or he will lose much of the educational value of the study.

Particular attention has been paid to the grading of the exercises.

The introductory chapter extends the familiar processes of arithmetic to the corresponding processes of algebra. The pupil is led by simple exercises, similar to those in arithmetic, to understand the use of letters to represent general and unknown numbers. Negative numbers are naturally introduced in connection with the extension of subtraction of arithmetical numbers. The meaning and use of positive and negative numbers, in the fundamental operations, are properly emphasized.

Equations and problems are distributed throughout the book. The importance of equivalent equations is not overlooked, but is very briefly and simply considered in Chapter IV. Until that chapter is reached, the solutions of equations should be *checked.*

All the matter in the book is printed in large type, and much pains has been taken to make the pages open and attractive.

Any suggestions from teachers and others will be greatly appreciated.

The authors have much pleasure in expressing their satisfaction with the excellence of the mechanical execution of the work, due to the ability and painstaking care of Messrs. J. S. Cushing & Co. and Messrs. Berwick & Smith, of the Norwood Press.

PREFACE TO THE SECOND EDITION.

In response to a demand for an edition of the Secondary Algebra containing chapters on subjects not included in the regular edition, the authors have issued such a book under the title *Complete Secondary Algebra.*

It has seemed advisable to include some of this additional matter in the second edition of the Secondary Algebra. This edition, therefore, differs from the first in having chapters on Permutations and Combinations, and Probability, and a fuller treatment of Limits and Infinite Series.

The *Complete Secondary Algebra* contains, in addition to the subjects treated in this book, chapters on Continued Fractions, Summation of Series, Exponential and Logarithmic Series, Determinants, and Theory of Equations.

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CHAPTER I.

INTRODUCTION.

GENERAL NUMBER.

1. Algebra, like Arithmetic, treats of number.

2. The examples

$$\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7} \text{ and } \frac{5}{11}+\frac{4}{11}=\frac{5+4}{11}=\frac{9}{11}$$

are particular cases of the following principle:

The sum of two fractions which have a common denominator is a fraction whose denominator is the common denominator, and whose numerator is the sum of the numerators; or, more briefly stated,

$$\frac{1st\ num.}{com.\ den.}+\frac{2d\ num.}{com.\ den.}=\frac{1st\ num.+2d\ num.}{com.\ den.}.$$

This principle can be stated still more concisely by letting letters stand for the two numerators and the common denominator.

Let a stand for 1*st num.*, b for 2*d num.*, and c for *com. den.* We then have

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}.$$

This relation states by means of letters, or symbols, all that is contained in the verbal statement. The letters a, b, and c stand for the terms of any two fractions, and therefore denote *any numbers whatever.*

In the first example above, $a=2$, $b=3$, $c=7$; in the second, $a=5$, $b=4$, $c=11$.

3. In ordinary Arithmetic all numbers are represented by the Arabic numerals, 1, 2, 3, etc. Each of these symbols stands for a definite number. The symbol 7, for instance, stands for a group of *seven* units, the symbol 5 for a group of *five* units.

But in Algebra, such symbols as a, b, x, y, are used to represent numbers which may have *any values whatever*, as in Art. 2.

For the sake of brevity we shall say *the number* a, or simply a, meaning thereby *the number denoted by the symbol* a.

4. The numbers represented by letters are, for the sake of distinction, called **Literal** or **General Numbers.**

EXERCISES I.

If p is the product obtained by multiplying a by b, express in symbols the following principles of multiplication:

1. The multiplicand is equal to the product divided by the multiplier. Let $p = 35$, $a = 7$, $b = 5$; $p = 24$, $a = 3$, $b = 8$.

2. The multiplier is equal to the product divided by the multiplicand. Let $p = 63$, $a = 9$, $b = 7$; $p = 40$, $a = 5$, $b = 8$.

If q is the quotient obtained by dividing m by n, express in symbols the following principles of division:

3. The dividend is equal to the divisor multiplied by the quotient. Let $q = 9$, $m = 99$, $n = 11$; $q = 6$, $m = 42$, $n = 7$.

4. The divisor is equal to the dividend divided by the quotient. Let $q = 5$, $m = 45$, $n = 9$; $q = 6$, $m = 72$, $n = 12$.

State in verbal language the principles which are expressed in symbols in the following:

5. $\dfrac{a}{c} - \dfrac{b}{c} = \dfrac{a-b}{c}$.

6. $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{a \times c}{b \times d}$.

7. $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c}$.

8. $\dfrac{a}{b} \times n = \dfrac{a \times n}{b}$.

9. $\dfrac{a}{d} = \dfrac{a \times n}{d \times n}$.

10. $\dfrac{a}{d} = \dfrac{a \div n}{d \div n}$.

11–16. In Exs. 5–10, let $a = 8$, $b = 7$, $c = 5$, $d = 6$, $n = 2$; $a = 15$, $b = 8$, $c = 11$, $d = 5$, $n = 5$.

5. As was assumed in Art. 2, the operations of Addition, Subtraction, Multiplication, and Division are denoted by the same symbols in Algebra as in Arithmetic.

Just as $5+3$, read *five plus three*, means that 3 is to be added to 5; so $a+b$, read *a plus b*, means that b is to be added to a.

Just as $5-3$, read *five minus three*, means that 3 is to be subtracted from 5; so $a-b$, read *a minus b*, means that b is to be subtracted from a.

Just as 5×3, read *five multiplied by three*, means that 5 is to be multiplied by 3; so $a\times b$, read *a multiplied by b*, means a is to be multiplied by b.

Just as $10\div 5$, read *ten divided by five*, means that 10 is to be divided by 5; so $a\div b$, read *a divided by b*, means that a is to be divided by b.

6. Since $5\times 3=3\times 5$, either number may be taken as the multiplier, the other as the multiplicand.

If the number on the left be taken as the multiplier, the symbol of multiplication is read *times* or *into*.

As, 5×3, read *five times three*, if 5 be regarded as the multiplier.

A dot ($\cdot$) is frequently used, instead of the symbol $\times$, to denote multiplication; as $a\cdot b$ for $a\times b$.

7. The symbol of multiplication between two literal numbers, or one literal number and an Arabic numeral, is frequently omitted.

E.g., the product $x\times y\times z$, or $x\cdot y\cdot z$, is usually written xyz, and is read *x-y-z*.

But the symbol of multiplication between two numerals cannot be omitted without changing the meaning.

E.g., if in the indicated multiplication, 3×6, or $3\cdot 6$, the symbol, $\times$, or $\cdot$, were omitted, we should have 36, not 18.

8. In a chain of additions and subtractions the operations are to be performed successively from left to right.

E.g., $$7+4-3+2=11-3+2=8+2=10.$$

In a chain of multiplications and divisions the operations are to be performed successively from left to right.

E.g., $12 \times 2 \div 3 \times 4 = 24 \div 3 \times 4 = 8 \times 4 = 32.$

In a chain of additions, subtractions, multiplications, and divisions, the multiplications and divisions are first to be performed, and then the additions and subtractions.

E.g., $2 \times 3 + 8 \div 4 = 6 + 2 = 8.$

9. An **Algebraic Expression** is a number expressed by means of the signs and symbols of Algebra; as x, mn, $ab - cd$, etc.

10. The **Symbol of Equality,** $=$, read *is equal to,* is placed between two numbers to indicate that they have the same or equal values; as $3 + 2 = 5$.

11. The **Symbol of Inequality,** $>$, read *is greater than,* is used to indicate that the number on its left is greater than that on its right; as $7 > 5$.

12. The **Symbol of Inequality,** $<$, read *is less than,* is used to indicate that the number on its left is less than that on its right; as $3 < 4 + 2$.

13. The use of letters to represent general numbers may be illustrated by a few simple examples.

Ex. 1. If a boy has 3 books and is given 2 more, he will have $3 + 2$ books. If he has a books and is given 5 more, he will have $a + 5$ books. If he has m books and is given n more, he will have $m + n$ books.

Ex. 2. If a man buys 5 city lots at 120 dollars each, he pays 120×5 dollars for the lots. If he buys a lots at 150 dollars each, he pays $150\,a$ dollars for the lots. If he buys u lots at v dollars each, he pays vu dollars for the lots.

Ex. 3. If a train runs 60 miles in two hours, it runs $60 \div 2$ miles in 1 hour. If it runs a miles in 5 hours, it runs $a \div 5$ miles in 1 hour. If it runs p miles in q hours, it runs $p \div q$ miles in 1 hour.

Ex. 4. If, in a number of *two* digits, the digit in the *units'* place is 3 and the digit in the *tens'* place is 5, the number is $10 \times 5 + 3$. If the digit in the units' place is a and the digit in the tens' place is b, the number is $10\,b + a$.

Ex. 5. Just as $2 = 1 + 1$, and $3 = 1 + 1 + 1$,

so $$2\,a = a + a, \text{ and } 3\,a = a + a + a.$$

Therefore, just as $3 + 2 = 5$, so $3\,a + 2\,a = 5\,a$.

In like manner, $$5\,x - 3\,x = 2\,x;$$

and $$\tfrac{1}{2}\,x + \tfrac{2}{3}\,x = \tfrac{7}{6}\,x.$$

EXERCISES II.

Read the following expressions:

1. $a + b$. **2.** $m - n$. **3.** $a \times b$. **4.** $m \div n$.

5. $4\,x + 2\,y$. **6.** $3\,m - 8\,n$. **7.** $4\,a \times 5\,b$. **8.** $7\,x \div 3\,y$.

9. $a + b + c$. **10.** $x - y + z$. **11.** $m - n - p$.

12. $4\,a - c + 3\,d$. **13.** $ab + bc - ac$. **14.** $3\,xy - 5\,bcd$.

15. A father is n years older than his son. How old is the father, if the son is 10 years old? If the son is x years old?

16. A boy rides his bicycle x miles and then walks y miles. How many miles does he go altogether?

17. A man has \$$d$. If he spends \$10, how many dollars has he left? If he spends \$$z$, how many dollars has he left?

18. A man is now n years old. How old was he 8 years ago? m years ago? How long must he live to be 75 years old? How long to be y years old?

19. If one pencil costs 3 cents, how much do 5 pencils cost? x pencils?

20. If one pencil costs c cents, how much do z pencils cost?

21. How many square feet are there in a floor 15 feet long and 20 feet wide? In a floor a feet long and b feet wide?

22. A train runs m miles in 1 hour. How many miles will it run in 4 hours? In b hours?

23. A train runs m miles in 8 hours. How many miles will it run in 1 hour? If it runs m miles in h hours, how many miles will it run in 1 hour?

24. A boy paid c cents for 5 note-books. How much did he pay for each? If he paid c cents for n note-books, how much did he pay for each?

25. In 3 dimes there are 10×3 cents. How many cents in d dimes? In x dimes?

26. How many cents in a dollars and b dimes? In x dollars, y dimes, and z cents?

27. 10×2, 10×3, 10×4, etc., are particular multiples of 10. Write *any* multiple of 10.

28. Write a number containing 8 units and 5 tens. Containing u units and t tens.

29. Write a number containing h hundreds, t tens, and u units. Containing a hundreds, b tens, and c units.

What are the values of the following expressions?

30. $a + a$.	**31.** $a + 2a$.	**32.** $x + 3x$.
33. $a - a$.	**34.** $2a - a$	**35.** $3z - z$.
36. $3c + 5c$.	**37.** $5d - 3d$.	**38.** $8x + 5x$.
39. $8x - 5x$.	**40.** $x + \frac{1}{3}x$.	**41.** $x - \frac{1}{3}x$.
42. $\frac{3}{4}a + \frac{1}{3}a$.	**43.** $\frac{3}{4}a - \frac{1}{3}a$.	**44.** $5m - \frac{5}{3}m$.
45. $a + 2a + 3a$.	**46.** $a + 2a - 3a$.	**47.** $5z + 8z + 4z$.
48. $8z - 5z + 4z$.	**49.** $9x + 3x - 8x$.	**50.** $9y - 4y - 3y$.

51. A man has $\$10x$. If he receives $\$8x$, how many dollars will he have? If he spends $\$6x$, how many dollars will he have left?

52. A boy paid $3x$ cents for pencils and $8x$ cents for note-books. How much did he pay for both? How much more for note-books than for pencils?

53. A girl has x dimes and $3x$ cents. How many cents has she?

54. A girl has a dollars. If she spends $7a$ dimes, how many dimes will she have left? If she spends $85a$ cents, how many cents will she have left?

55. A man has \$$45x$. If he spends \$$7x$ for a lot, and \$$32x$ for a house, how many dollars will he have left?

56. A boy rides a wheel x miles and then walks $160x$ rods. How many rods did he go altogether? How many rods more did he ride than walk?

57. The width of a room is x yards, and the length is $2x$ feet greater than the width. How many feet are there in the length of the room?

Axioms.

14. An **Axiom** is a truth so simple that it cannot be made to depend upon a truth still simpler.

Algebra makes use of the following mathematical axioms:

(i.) *Every number is equal to itself.* *E.g.*, $7 = 7$, $a = a$.

(ii.) *The whole is equal to the sum of all its parts.*

E.g., $7 = 3 + 4$, $5 = 1 + 1 + 1 + 1 + 1$.

(iii.) *If two numbers be equal, either can replace the other in any algebraic expression in which it occurs.*

E.g., If $a + b = c$, and $b = 2$, then $a + 2 = c$, replacing b by 2.

(iv.) *Two numbers which are each equal to a third number are equal to each other.*

E.g., If $a = b$, and $c = b$, then $a = c$.

(v.) *The whole is greater than any of its parts; and, conversely, any part is less than the whole.*

E.g., $3 + 2 > 2$ and $2 < 3 + 2$.

15. *Literal numbers*, as has been stated, are numbers which may have *any values whatever*. But it is frequently necessary to assign particular values to such numbers.

16. **Substitution** is the process of replacing a literal number in an algebraic expression by a particular value. See axiom (iii.). Simple examples in substitution have already been given in Art. 2.

Ex. 1. If, in $a + b$, we let $a = 3$ and $b = 5$, then

$$a + b = 3 + 5 = 8, \text{ or } a + b = 8.$$

Ex. 2. If, in $a + b - 2a + 3b - c$, we let $a = 6$, $b = 11$, $c = 1$, we have

$$\begin{aligned} a + b - 2a + 3b - c &= 6 + 11 - 2 \times 6 + 3 \times 11 - 1 \\ &= 6 + 11 - 12 + 33 - 1 = 37. \end{aligned}$$

Ex. 3. If, in the last example, $a = 3$, $b = 1$, and $c = 1$, we have $a + b - 2a + 3b - c = 3 + 1 - 6 + 3 - 1 = 4 - 6 + 3 - 1$.

We cannot further reduce $4-6+3-1$, since we are unable, *as yet*, to subtract 6 from 4.

EXERCISES III.

When $a = 10$, $b = 5$, $c = 3$, find the values of the following expressions:

1. $a + b$.
2. $a - b$.
3. ab.
4. $a \div b$.
5. $a + b - c$.
6. $a - b + c$.
7. $a - b - c$.
8. $c + 3a$.
9. $5b - 3c$.
10. $2a + 3b - 5c$.
11. $5a - 2b - 6c$.
12. $3a - 5b + 8c$.
13. $7ab$.
14. $2abc$.
15. $3abb$.
16. $2ab + 5ac$.
17. $3ac - 5bc$.
18. $5aa - 3bb$.
19. $2ab - 3ac + 5bc$.
20. $5aa - 3bb + 6cc$.

Fundamental Principles.

17. The following principles are obtained directly from the axioms:

(i.) *If the same number, or equal numbers, be added to equal numbers, the sums will be equal.*

(ii.) *If the same number, or equal numbers, be subtracted from equal numbers, the remainders will be equal.*

(iii.) *If equal numbers be multiplied by the same number, or by equal numbers, the products will be equal.*

(iv.) *If equal numbers be divided by the same number (except* 0), *or by equal numbers, the quotients will be equal.*

E.g., if $3x = 6$,

then $3x + 2 = 6 + 2,\quad 3x - 5 = 6 - 5,$

$$3x \times 4 = 6 \times 4,\quad 3x \div 3 = 6 \div 3.$$

Equations.

18. An **Equation** is a statement that two expressions are equal; as $7 \times 9 = 63$, $4 \times 7 + 3 = 31$.

The **First Member** of an equation is the expression on the *left* of the symbol $=$; the **Second Member** is the expression on the *right* of the symbol $=$.

19. Ex. 1. What is the value of x in the equation

$$3x + 8x = 22\,?$$

Since $3x + 8x = 11x$, we have

$$11x = 22.$$

Dividing both members by 11 [Art. 17, (iv.)],

$$x = 2.$$

To check this result we substitute 2 for x in the equation

$$3x + 8x = 3 \times 2 + 8 \times 2 = 6 + 16 = 22.$$

Ex. 2. If $8x - 3x$ has the value 20, what is the value of x?

We have $8x - 3x = 20.$

Or, since $8x - 3x = 5x,\quad 5x = 20.$

Dividing both members by 5, $x = 4.$

Check: $8 \times 4 - 3 \times 4 = 32 - 12 = 20.$

20. An **Unknown Number** of an equation is a number whose value is to be found from the equation.

The **Known Numbers** of an equation are the numbers whose values are given.

In the equation $$x + 1 = 3,$$

the unknown number is x, and the known numbers are 1 and 3.

Unknown numbers are usually represented by the final letters of the alphabet, x, y, z, etc., as in the above examples.

EXERCISES IV.

Find the value of x in each of the following equations:

1. $3x = 9$. **2.** $6x = 18$. **3.** $5x = 0$.

4. $\frac{1}{3}x = 4$. **5.** $\frac{1}{4}x = 5$. **6.** $\frac{1}{2}x = 0$.

7. $\frac{2}{3}x = 6$. **8.** $\frac{5}{6}x = 15$. **9.** $\frac{7}{8}x = 21$.

10. $x + x = 8$. **11.** $x + 5x = 24$. **12.** $5x + 4x = 45$.

13. $5x - 4x = 3$. **14.** $6x - 3x = 9$. **15.** $7x - 5x = 12$.

16. $x + 3x + 5x = 18$. **17.** $2x + 5x + 3x = 20$.

18. $7x + 3x + 5x = 90$. **19.** $5x + 4x - 6x = 15$.

20. $8x - 5x + x = 12$. **21.** $11x + 7x - 5x = 26$.

22. $x + \frac{1}{2}x = 6$. **23.** $x - \frac{1}{2}x = 10$.

24. $24x + \frac{5}{6}x = 149$. **25.** $3x + \frac{3}{4}x = 30$.

26. $5x - \frac{7}{8}x = 33$. **27.** $2\frac{1}{2}x - \frac{1}{6}x = 14$.

28. $x + \frac{1}{2}x + \frac{5}{6}x = 28$. **29.** $2x - \frac{1}{2}x + \frac{5}{8}x = 34$.

30. $\frac{3}{4}x + \frac{5}{7}x - \frac{1}{2}x = 54$. **31.** $5x - \frac{2}{3}x - \frac{1}{5}x = 62$.

Problems solved by Equations.

21. A **Problem** is a question proposed for solution.

Another use of literal numbers is shown by the following problems:

Pr. **1.** The older of two brothers has twice as many marbles as the younger, and together they have 33 marbles. How many has the younger?

The number of marbles the younger brother has is, as yet, an *unknown number.*

Let us represent this unknown number by some letter, say x.

Then, since the older brother has twice as many, he has $2x$ marbles.

The problem states,

in *verbal* language: *the number of marbles the younger has plus the number the older has is equal to* 33;

in *algebraic* language, $x + 2x = 33$,

or, $3x = 33$.

Dividing both members of the last equation by 3, we have

$$x = 11,$$

the number of marbles the younger has.

The older has, $2x, = 2 \times 11, = 22$ marbles.

To check this result, we substitute 11 for x in the equation of the problem:

$$x + 2x = 11 + 22 = 33.$$

Notice that the letter x stands for an abstract number. The beginner must never put x for marbles, distance, time, etc., but for the *number* of marbles, of miles, of hours, etc.

Pr. 2. Divide 52 into three parts, so that the second shall be one-half of the first, and the third one-fourth of the second.

Let x stand for the first part.

Then $\frac{1}{2}x$ stands for the second part,

and $\frac{1}{4} \times \frac{1}{2}x, = \frac{1}{8}x$, stands for the third part.

The problem states,

in *verbal* language: *the first part, plus the second part, plus the third part, is equal to* 52;

in *algebraic* language, $x + \frac{1}{2}x + \frac{1}{8}x = 52$,

or, $\frac{13}{8}x = 52$.

Dividing both members of the last equation by 13,

$$\tfrac{1}{8}x = 4.$$

Multiplying both members of this equation by 8,

$$x = 32,$$

the first part. Then the second part is

$$\tfrac{1}{2}x, \ = \tfrac{1}{2} \times 32, \ = 16,$$

and the third part is

$$\tfrac{1}{8}x, \ = \tfrac{1}{8} \times 32, \ = 4.$$

Check: $x + \tfrac{1}{2}x + \tfrac{1}{8}x = 32 + 16 + 4 = 52.$

22. In stating problems in algebraic language, the beginner should observe the following directions:

(i.) *Read the problem carefully, and note what are the numbers whose values are required.*

(ii.) *Let some letter, say x, stand for one of the required numbers.*

(iii.) *The problem will contain statements about the values of other numbers. Use these statements to express their values in terms of x.*

(iv.) *Express concisely in verbal language a statement in the problem which furnishes an equation.*

(v.) *Express this statement in algebraic language.*

EXERCISES V.

1. What number is five times x? Twelve times x?

2. Five times a number is 80. What is the number?

3. Twelve times a number is 132. What is the number?

4. The greater of two numbers is four times the less. If the less is x, what is the greater? What is their sum? Their difference?

5. The greater of two numbers is four times the less. If their sum is 75, what are the numbers?

6. The greater of two numbers is seven times the less. If their difference is 72, what are the numbers?

7. A father is three times as old as his son. If the son is x years old, how old is the father? What is the sum of their ages? How much older is the father than the son?

8. A father is three times as old as his son, and the sum of their ages is 48 years. How old is each?

9. A father is five times as old as his son. If the father is 32 years older than his son, what are their ages?

10. At an election A received twice as many votes as B, and his majority was 138. How many votes did each receive?

11. In a company are 32 persons. The number of children is three times the number of adults. How many are there of each?

12. Two trains leave Philadelphia in opposite directions. After one hour they are 60 miles apart. If one has gone three times as far as the other, how many miles is each from Philadelphia?

13. Two trains leave Chicago in the same direction. After one hour they are 20 miles apart. If one has gone twice as far as the other, how far is each from Chicago?

14. A man pays $ 55 in one-dollar bills and ten-dollar bills. If he pays the same number of one-dollar bills as of ten-dollar bills, how many of each does he pay?

15. In a number of two digits, the tens' digit is three times the units' digit, and their sum is 8. What are the digits? What is the number?

16. In a number of two digits, the units' digit is twice the tens' digit, and their difference is 3. What is the number?

17. What is the sum of twice x and six times x? The difference?

18. If twice a number is added to six times the same number, the sum will be 96. What is the number?

19. If four times a number is subtracted from seven times the same number, the remainder will be 72. What is the number?

20. A traveller first rides his bicycle 9 miles an hour. He then rides the same number of hours in a car 35 miles an hour. If he travels 132 miles, how many hours did he ride his bicycle?

21. Two trains run out of New York in opposite directions. One runs 42 miles an hour, the other 34 miles an hour. After how many hours will they be 228 miles apart?

22. Two trains run out of New York in the same direction. One runs 38 miles an hour, the other 34 miles an hour. After how many hours will they be 32 miles apart?

23. A boy has 75 cents in dimes and five-cent pieces. He has the same number of dimes as of five-cent pieces. How many coins of each kind has he?

24. A owes B $40. He pays his debt in ten-dollar bills, and receives in change the same number of two-dollar bills. How many ten-dollar bills did A pay B?

25. A cistern has two pipes. One lets in 8 gallons a minute, and the other 15 gallons a minute. If the cistern holds 207 gallons, how many minutes will it take the pipes to fill it?

26. A cistern has two pipes. One lets in 11 gallons a minute, and the other lets out 6 gallons a minute. How many minutes will it take the one pipe to let in 85 gallons more than the other lets out?

27. What is the sum of x, four times x, and seven times x? Of x, twice x, and five times x?

28. The sum of a certain number, four times the number, and seven times the number is 96. What is the number?

29. Three boys, A, B, and C, together have 21 pencils. B has twice as many as A, and C four times as many as A. How many has A? How many has each?

30. Divide 147 into three parts, so that the second part shall be four times the first, and the third part twice the first.

31. A merchant receives \$ 64 in ten-dollar bills, five-dollar bills, and one-dollar bills. He receives the same number of each kind. How many of each does he receive?

32. At an election 726 votes were cast. A, B, and C were candidates. B received three times as many votes as C, and A twice as many as C. How many votes did each receive?

33. A cistern has three pipes. The first lets in 6 gallons a minute, the second 9 gallons a minute, and the third 12 gallons a minute. If the cistern holds 243 gallons, how long will it take the pipes to fill it?

34. A cistern has three pipes. The first lets in 5 gallons a minute, the second 14 gallons a minute, and the third lets out 10 gallons a minute. How many minutes will it take the two pipes to let in 108 gallons more than the third pipe lets out?

35. An estate of \$ 9600 is divided among 2 sons and 2 daughters. The sons receive equal amounts, and a daughter receives three times as much as a son. How many dollars does each receive?

36. What is twice $3x$? Seven times $5x$? Four times $9x$?

37. A receives x dollars, B receives three times as much as A, and C receives twice as much as B. How many dollars does C receive? How many dollars do all receive?

38. Three boys, A, B, and C, together receive \$ 70. B receives three times as much as A, and C twice as much as B. How many dollars does each receive?

39. A merchant's profits doubled each year for three years. If his profits for the three years were \$ 8750, what were his profits the first year?

40. In a company are 50 persons. The number of women is three times the number of men, and the number of children is twice the number of women. How many of each are in the company?

41. What number is $\frac{1}{4}$ *of* x? $\frac{3}{2}$ of x?

42. If $\frac{1}{4}$ of a number is 16, what is the number?

43. The less of two numbers is $\frac{3}{4}$ of the greater. If the greater is x, what is the less? What is their sum? Their difference?

44. The less of two numbers is $\frac{3}{4}$ of the greater. If their sum is 91, what are the numbers?

45. A and B together have \$1133. If B has $\frac{4}{7}$ as much as A, how many dollars has each?

46. A has \$31 more than B. If B has $\frac{3}{4}$ as much as A, how many dollars has each?

47. Two boys, A and B, catch 36 fish. If A catches $\frac{4}{5}$ as many as B, how many fish does each catch?

48. A workman pays $\frac{3}{7}$ of his wages for board. If he has left \$8 each week, what are his wages?

49. Two boys together solve 65 problems. If the first solves $\frac{5}{8}$ as many as the second, how many problems does each solve?

50. A solves 21 more problems than B. If B solves $\frac{2}{5}$ as many as A, how many problems does each solve?

51. A tree 126 feet high is broken by the wind. If the part left standing is $\frac{3}{11}$ of the part broken off, how long is each part?

52. What is the sum of $\frac{1}{3}$ of x and $\frac{3}{4}$ of x? The difference?

53. If $\frac{1}{3}$ of a number is added to $\frac{3}{4}$ of the same number, the sum will be 39. What is the number?

54. If $\frac{3}{5}$ of a number is subtracted from $\frac{3}{4}$ of the same number, the remainder will be 3. What is the number?

55. If to a number is added $\frac{1}{3}$ of itself and $\frac{3}{4}$ of itself, the sum will be 50. What is the number?

56. Three boys, A, B, and C, together have 29 pencils. B has $\frac{2}{3}$ as many as A, and C has $\frac{3}{4}$ as many as A. How many pencils has each?

57. Divide 104 into three parts, so that the first shall be three times the second, and the third $\frac{1}{3}$ of the second.

58. A man makes a journey of 69 miles. He goes $\frac{3}{5}$ as far by boat as by train, and $\frac{1}{8}$ as far by stage as by train. How many miles does he go by each conveyance?

59. What is $\frac{1}{5}$ of three times x? Twice $\frac{2}{3}$ of x? $\frac{3}{4}$ of $\frac{5}{2}$ of x?

60. The second of three numbers is three times the first, and the third is $\frac{1}{5}$ of the second. If the first number is x, what is the second? The third? What is the sum of the three numbers?

61. The sum of three numbers is 99. The second is four times the first, and the third is $\frac{2}{5}$ of the second. What are the numbers?

62. The width of a field is $\frac{4}{7}$ of its length, and the distance around it is 88 rods. What is the width and the length of the field?

63. The sum of $420 is divided among A, B, and C. B receives $\frac{2}{3}$ as much as A, and C as much as A and B together. How many dollars does each receive?

64. A sells a number of apples at 2 cents apiece, and B sells $\frac{7}{4}$ as many at 3 cents apiece. If they receive together 87 cents, how many apples does each sell?

Parentheses.

23. Parentheses, ( ), and **Brackets,** [ ], are used to indicate that whatever is placed within them is to be treated as a whole.

E.g., $10 - (2 + 5)$ means that the result of adding 5 to 2, or 7, is to be subtracted from 10; that is,

$$10 - (2 + 5) = 10 - 7 = 3.$$

But $10 - 2 + 5$ means that 2 is to be subtracted from 10 and 5 is then to be added to that result; that is,

$$10 - 2 + 5 = 8 + 5 = 13.$$

In like manner, $[27-(3+2)\times 5]\div 2$ means that the result of multiplying the sum $3+2$ by 5 is first to be subtracted from 27, and the remainder is then to be divided by 2; that is,

$$[27-(3+2)\times 5]\div 2=[27-25]\div 2=2\div 2=1.$$

Likewise, $(a+b)c$ is the result of multiplying $a+b$ by c, etc.

EXERCISES VI.

Find the value of each of the following expressions:

1. $10+(3+2)$. **2.** $10-(3+2)$. **3.** $10+(3-2)$.

4. $10-(3-2)$. **5.** $27-(18-11)$. **6.** $53+(40+7)$.

7. $97+(11-8)$. **8.** $58-(15-7)$. **9.** $99+(18-17)$.

10. $5(8+2)$. **11.** $6(11-6)$. **12.** $(10+15)\div 5$.

13. $10+(15\div 5)$. **14.** $(12-4)\div 2$. **15.** $12-(4\div 2)$.

16. $(15-3)+(18-6)$. **17.** $(16-2)-(20-8)$.

18. $(4+5)(8-3)$. **19.** $(8+12)\div(7-2)$.

20. $20+[11-(5+2)]$. **21.** $28-[16-(5+3)]$.

22. $[26-(14+6)]\times 5$. **23.** $[27-(18-12)]\div 7$.

When $a=12$, $b=6$, $c=3$, find the values of:

24. $a+(b-c)$. **25.** $a-(b+c)$. **26.** $a-(b-c)$.

27. $c+5(a-b)$. **28.** $4a-2(b+c)$. **29.** $b[c+(a-b)]$.

30. $a[a-\frac{1}{3}(b+c)]$. **31.** $[a-(b-c)]\div c$. **32.** $[b+(a-c)]\div c$.

POSITIVE AND NEGATIVE NUMBERS, OR ALGEBRAIC NUMBERS.

24. In ordinary Arithmetic we subtract a number from an equal or a greater number. We are familiar with such operations as

$$\begin{array}{cccl} 5 & 4 & 3 & \text{minuend} \\ \underline{3} & \underline{3} & \underline{3} & \text{subtrahend} \\ 2 & 1 & 0 & \text{remainder} \end{array} \qquad \text{(i.)}$$

But such operations as

2	1	0	minuend	
3	3	3	subtrahend	(ii.)
?	?	?	remainder	

have not occurred in ordinary Arithmetic. In Arithmetic we cannot subtract from a number more units than are contained in the number.

Now, as the minuend in (i.) decreases, the remainder decreases. When the minuend is equal to the subtrahend, the remainder is 0. If then, as in (ii.), the minuend become less than the subtrahend, the remainder must become less than 0.

The operation of subtracting a greater number from a less is therefore possible only when numbers less than zero are assumed.

25. *Numbers less than zero* are called **Negative Numbers.** *Numbers greater than zero* are, for the sake of distinction, called **Positive Numbers.**

Positive and negative numbers are called **Algebraic** or **Relative Numbers.**

A *positive* number may be indicated by placing a small sign, $^+$, before the symbols for *one, two, three,* etc.; as $^+1$, $^+2$, $^+3$, etc., read *positive one, positive two, positive three,* etc.

A *negative* number may be indicated by placing a small sign, $^-$, before the symbols for *one, two, three,* etc.; as $^-1$, $^-2$, $^-3$, etc., read *negative one, negative two, negative three,* etc.

We can now write (i.) and (ii.) as follows:

$^+5$	$^+4$	$^+3$	$^+2$	$^+1$	0	min.	
$^+3$	$^+3$	$^+3$	$^+3$	$^+3$	$^+3$	sub.	(iii.)
$^+2$	$^+1$	0	$^-1$	$^-2$	$^-3$	rem.	

26. From the preceding article we have:

Zero *is the result of subtracting a number from an equal number.*

E.g., $0 = {^+7} - {^+7} = {^-5} - {^-5} = {^+n} - {^+n} = {^-n} - {^-n}$.

27. We thus have in Algebra the series of numbers,

$$\cdots, {}^{+}5, {}^{+}4, {}^{+}3, {}^{+}2, {}^{+}1, 0, {}^{-}1, {}^{-}2, {}^{-}3, {}^{-}4, {}^{-}5, \cdots,$$

wherein the signs, $\cdots$, indicate that the succession of numbers continues without end in both directions. This series is usually written with the positive numbers on the right, as

$$\cdots, {}^{-}5, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, {}^{+}4, {}^{+}5, \cdots.$$

28. In this series the numbers increase by *one* from left to right, and decrease by *one* from right to left. Or, a number is greater than any number on its left, and less than any number on its right.

Thus, ${}^{+}2$ is one unit greater than ${}^{+}1$, two units greater than 0, three units greater than ${}^{-}1$, etc. Again, ${}^{-}3$ is three units greater than ${}^{-}6$, two units less than ${}^{-}1$, three units less than 0, etc.

29. The signs ${}^{+}$ and ${}^{-}$ are called signs of *quality;* the signs $+$ and $-$, signs of *operation.* The two sets of signs must, as yet, be carefully distinguished.

30. The **Absolute Value** of a number is the number of units contained in it without regard to their *quality.*

E.g., the absolute value of ${}^{+}4$ is 4, of ${}^{-}5$ is 5.

31. From the results of the preceding articles, we obtain the following general relations:

(i.) *Of two positive numbers, that number is the greater which has the greater absolute value; and that number is the less which has the less absolute value.*

(ii.) *Of two negative numbers, that number is the greater which has the less absolute value; and that number is the less which has the greater absolute value.*

For example, ${}^{-}3 > {}^{-}5$, or ${}^{-}5 < {}^{-}3$, since ${}^{-}5$ is five units less than 0, and ${}^{-}3$ is only three units less than 0.

32. It is important to notice that a negative remainder does not mean that more units have been taken from the minuend than were contained in it; *such a remainder indicates that the subtrahend is greater than the minuend by as many units as are contained in the remainder.*

Thus, in $^{+}15 - {}^{+}25 = {}^{-}10$, the remainder, $^{-}10$, indicates that the subtrahend is 10 units greater than the minuend.

33. It is evidently necessary thus to enlarge the meaning of subtraction in such an expression as $a - b$. For, if a and b are to have any values whatever, the case in which b is greater than a, that is, in which *the subtrahend is greater than the minuend,* must be included in the operation of subtraction.

34. Negative numbers have been introduced by extending the operation of subtraction. But it is necessary to treat them as numbers apart from this particular operation.

As in Arithmetic, so in Algebra, any integer is an aggregate of like units.

Just as $$4 = 1 + 1 + 1 + 1,$$

so $$^{+}4 = {}^{+}1 + {}^{+}1 + {}^{+}1 + {}^{+}1, \text{ and } {}^{-}4 = {}^{-}1 + {}^{-}1 + {}^{-}1 + {}^{-}1.$$

Just as $$\tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3},$$

so $$^{+}(\tfrac{2}{3}) = {}^{+}(\tfrac{1}{3}) + {}^{+}(\tfrac{1}{3}), \text{ and } {}^{-}(\tfrac{2}{3}) = {}^{-}(\tfrac{1}{3}) + {}^{-}(\tfrac{1}{3}).$$

EXERCISES VII.

Simplify the following expressions:

1. $^{+}17 - {}^{+}4$. **2.** $^{+}17 - {}^{+}17$. **3.** $^{+}17 - {}^{+}27$.

4. $^{+}25 - {}^{+}18$. **5.** $^{+}25 - {}^{+}25$. **6.** $^{+}25 - {}^{+}35$.

7. $^{+}88 - {}^{+}95$. **8.** $^{+}56 - {}^{+}27$. **9.** $^{+}101 - {}^{+}105$.

What value of x will make the first member of each of the following equations the same as the second?

10. $x - {}^{+}5 = {}^{+}7$. **11.** $x - {}^{+}5 = 0$. **12.** $x - {}^{+}5 = {}^{-}2$.

13. $x - {}^{+}11 = {}^{+}9$. **14.** $x - {}^{+}15 = {}^{-}13$. **15.** $x - {}^{+}12 = {}^{-}10$

How many units is each of the following numbers greater or less than 0?

16. $^{+}5$. **17.** $^{-}3$. **18.** $^{-}11$. **19.** $^{+}14$. **20.** $^{-}20$.

Which of each of the following pairs of numbers is the greater, and by how many units?

21. $^{+}7, ^{+}4$. **22.** $^{-}7, ^{-}4$. **23.** $^{+}7, ^{-}4$. **24.** $^{-}7, ^{+}4$.

25. $^{+}19, ^{+}9$. **26.** $^{-}29, ^{+}1$. **27.** $^{+}32, ^{-}3$. **28.** $^{-}14, ^{-}3$.

Positive and Negative Numbers are Opposite Numbers.

35. In Arithmetic we have: *the remainder added to the subtrahend is equal to the minuend.* This principle, like all principles of Arithmetic, is retained in Algebra. We therefore have from (iii.) Art. 25:

$$\begin{array}{rrrrrrl} ^{+}3 & ^{+}3 & ^{+}3 & ^{+}3 & ^{+}3 & ^{+}3 & \text{subtr.} \\ \underline{^{+}2} & \underline{^{+}1} & \underline{0} & \underline{^{-}1} & \underline{^{-}2} & \underline{^{-}3} & \text{rem.} \\ ^{+}5 & ^{+}4 & ^{+}3 & ^{+}2 & ^{+}1 & 0 & \text{min.} \end{array}$$

36. The equation $^{+}3 + {}^{-}3 = 0$ gives us the following important principle:

The sum of a positive number and a negative number having the same absolute value is equal to zero; i.e., *two such numbers cancel each other when united by addition.*

E.g., $^{+}1 + {}^{-}1 = 0$, $^{+}3 + {}^{-}3 = 0$, $^{-}17\frac{1}{2} + {}^{+}17\frac{1}{2} = 0$.

In general, $^{+}n + {}^{-}n = 0$.

For this reason, positive and negative numbers in their relation to each other are called *opposite* numbers. When their absolute values are equal, they are called *equal* and *opposite* numbers.

37. Any *quantities* which in their relation to each other are *opposite*, may be represented in Algebra by *positive* and *negative* numbers; as *credits* and *debits*, *gain* and *loss*.

Ex. 1. 100 *dollars credit* and 100 *dollars debit* cancel each other. That is, 100 *dollars credit united with* 100 *dollars debit is equal to neither credit nor debit;* or,

100 *dollars credit* + 100 *dollars debit* = *neither credit nor debit.*

If *credits* be taken *positively* and *debits negatively*, then 100 *dollars credit* may be represented by $^{+}100$, and 100 *dollars debit* by $^{-}100$. Their united effect, as stated above, may then be represented algebraically thus:

$$^{+}100 + {}^{-}100 = 0.$$

The result, 0, means *neither credit nor debit.*

Similarly for *opposite temperatures.*

Ex. 2. If a body is first heated 10° and then cooled down 8°, its final temperature is 2° above its original temperature; or, stated algebraically,

$$^{+}10 + {}^{-}8 = {}^{+}2.$$

The result, $^{+}2$, means a *rise* of 2° in temperature.

EXERCISES VIII.

Express algebraically each one of the following statements:

1. $45 gain and $45 loss is equivalent to neither gain nor loss.

2. $95 gain and $50 loss is equivalent to $45 gain.

3. $37 gain and $57 loss is equivalent to $20 loss.

4. If a man travels 220 miles due west and then 220 miles due east, he is at his starting place.

5. If a man ascends 2250 feet in a balloon and then descends 200 feet, he is 2050 feet above the earth.

6. If a man walks 90 feet to the right and then 110 feet to the left, he is 20 feet to the left of his starting point.

7. A rise of 20° in temperature, followed by a fall of 27°, is equivalent to a fall of 7°.

8. A rise of 15° in temperature, followed by a fall of 12°, is equivalent to a rise of 3°.

CHAPTER II.

THE FOUR FUNDAMENTAL OPERATIONS WITH ALGEBRAIC NUMBER.

ADDITION OF ALGEBRAIC NUMBERS.

1. *The* **Addition** *of two numbers is the process of uniting them into one aggregate.*

The numbers to be added are called **Summands.**

Addition of Numbers with Like Signs.

2. Ex. **1.** Add $^+3$ to $^+4$.

The three positive units, $^+3$, when added to the four positive units, $^+4$, give an aggregate of *four plus three,* or *seven,* positive units. That is,

$$^+4 + {}^+3 = {}^+(4+3) = {}^+7.$$

In like manner,

Ex. **2.** $^-4 + {}^-3 = {}^-(4+3) = {}^-7.$

These examples illustrate the following method of adding two numbers with like signs:

Add arithmetically their absolute values, and prefix to the sum their common sign of quality.

Addition of Numbers with Unlike Signs.

3. Ex. **1.** Add $^-2$ to $^+5$.

The two negative units, $^-2$, when added to the five positive units, $^+5$, *cancel two of the five positive units.* There remain then *five minus two,* or *three,* positive units. That is,

$$^+5 + {}^-2 = {}^+(5-2) = {}^+3.$$

Ex. 2. Add $^{+}2$ to $^{-}5$.

The two positive units, $^{+}2$, when added to the five negative units, $^{-}5$, *cancel two of the five negative units.* There remain then *five minus two,* or *three,* negative units. That is,

$$^{-}5 + {}^{+}2 = {}^{-}(5 - 2) = {}^{-}3.$$

Observe that in both examples *the sum is of the same quality as the number which has the greater absolute value. Also, that the absolute value of the sum is obtained by subtracting the less absolute value,* 2, *from the greater,* 5.

These examples illustrate the following method of adding two numbers with unlike signs:

Subtract arithmetically the less absolute value from the greater. To that remainder prefix the sign of quality of the number which has the greater absolute value.

The examples given in Ch. I, Art. 37, are concrete illustrations of the preceding principles.

4. Observe that a *positive* number *increases* a number to which it is added, while a *negative* number *decreases* it.

EXERCISES I.

Add:

1.	2.	3.	4.	5.	6.
$^{+}2$	$^{-}4$	$^{+}9$	$^{-}8$	$^{+}13$	$^{-}21$
$^{+}6$	$^{-}5$	$^{+}3$	$^{-}7$	$^{+}19$	$^{-}15$

7.	8.	9.	10.	11.	12.
$^{+}8$	$^{-}8$	$^{-}7$	$^{+}13$	$^{-}21$	$^{+}37$
$^{-}3$	$^{+}3$	$^{+}4$	$^{-}17$	$^{+}32$	$^{-}22$

SUBTRACTION OF ALGEBRAIC NUMBERS.

5. Subtraction *is the inverse of addition.* In addition two numbers are given, and it is required to find their sum, as in

$$^{+}9 + {}^{+}2 = {}^{+}11.$$

In subtraction the sum of two numbers and one of them are given, and it is required to find the other number, as in

$$^{+}11 - {}^{+}2 = ({}^{+}9 + {}^{+}2) - {}^{+}2 = {}^{+}9.$$

That is, *if from the sum of two numbers either of the numbers be subtracted, the remainder is the other number.*

In general, $$(a + b) - a = b.$$

6. Ex. 1. A man's net profits last year were 1200 dollars. This year his income is 150 dollars less, and his expenditures are the same. What are his net profits this year?

To take away 150 *dollars income is equivalent to adding* 150 *dollars expenditures.*

If net profits and income be taken positively, and expenditures negatively, the last statement, expressed algebraically, is

$$^{+}1200 - {}^{+}150 = {}^{+}1200 + {}^{-}150.$$

Ex. 2. A man's net profits last year were 1200 dollars. This year his income is the same and his expenditures are 150 dollars less. What are his net profits this year?

To take away 150 *dollars expenditures is equivalent to adding* 150 *dollars profits.*

The algebraic statement of this relation is

$$^{+}1200 - {}^{-}150 = {}^{+}1200 + {}^{+}150.$$

These examples illustrate the following principle:

To subtract one number from another number, reverse the sign of quality of the subtrahend, and add.

E.g., $^{+}2 - {}^{+}3 = {}^{+}2 + {}^{-}3, = {}^{-}1.$ $\qquad$ $^{-}2 - {}^{+}3 = {}^{-}2 + {}^{-}3 = {}^{-}5.$

$^{+}2 - {}^{-}3 = {}^{+}2 + {}^{+}3, = {}^{+}5.$ $\qquad$ $^{-}2 - {}^{-}3 = {}^{-}2 + {}^{+}3 = {}^{+}1.$

7. It is important to notice that the preceding examples do not prove this principle. The following examples illustrate a method of proof which may be used.

Ex. 1. Subtract $^+5$ from $^+7$.

In $^+7 - {}^+5$, the minuend, $^+7$, is to be expressed as the sum of two numbers, *one of which is* $^+5$. Since $^-5 + {}^+5 = 0$, we may write

$$^+7 = {}^+7 + {}^-5 + {}^+5 = ({}^+7 + {}^-5) + {}^+5.$$

That is, $^+7$ may be regarded as the sum of two numbers, one of which is $^+7 + {}^-5$, and the other is $^+5$. Therefore, by definition of subtraction,

$$\begin{aligned} ^+7 - {}^+5 &= [({}^+7 + {}^-5) + {}^+5] - {}^+5 \\ &= {}^+7 + {}^-5 = {}^+2, \end{aligned}$$

That is, *to subtract* $^+5$ *is equivalent to adding* $^-5$.

Ex. 2. Subtract $^-5$ from $^+7$.

We have
$$\begin{aligned} ^+7 - {}^-5 &= [({}^+7 + {}^+5) + {}^-5] - {}^-5 \\ &= {}^+7 + {}^+5 = {}^+12, \end{aligned}$$

That is, *to subtract* $^-5$ *is equivalent to adding* $^+5$.

8. We thus see that every operation of subtraction is equivalent to an operation of addition. On this account it is convenient to speak of a chain of additions and subtractions as an **Algebraic Sum.**

EXERCISES II.

Subtract:

1.	2.	3.	4.	5.	6.
$^+9$	$^+2$	$^+8$	$^+3$	$^-9$	$^-4$
$^+2$	$^+9$	$^+3$	$^+8$	$^-4$	$^-9$
7.	**8.**	**9.**	**10.**	**11.**	**12.**
$^-8$	$^-7$	$^+5$	$^-6$	$^+6$	$^-6$
$^-7$	$^-8$	$^+5$	$^-6$	$^-9$	$^+9$

MULTIPLICATION OF ALGEBRAIC NUMBERS.

9. In multiplication, the multiplicand and multiplier are called **Factors** of the product.

10. In ordinary Arithmetic, multiplication by an integer is defined as an abbreviated addition. Thus,

$$4 \times 3 = 4 + 4 + 4;$$

that is, the number 4 is taken three times as a summand.

But $$3 = 1 + 1 + 1.$$

We thus see that the product 4×3 is obtained from 4 just as 3 is obtained from the positive unit, 1.

We are thus naturally led to the following definition of multiplication:

The product is obtained from the multiplicand just as the multiplier is obtained from the positive unit.

11. The above definition is an extension of the meaning of arithmetical multiplication when the multiplier is an integer, and gives an intelligible meaning to arithmetical multiplication when the multiplier is a fraction.

Thus, $\frac{2}{3}$ is obtained from the unit, 1, by taking one-third of the latter twice as a summand; or

$$\tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3}.$$

In like manner, to multiply 5 by $\frac{2}{3}$, we take one-third of 5 twice as a summand; or

$$5 \times \tfrac{2}{3} = \tfrac{5}{3} + \tfrac{5}{3} = \tfrac{10}{3}.$$

12. There are two cases to be considered in the multiplication of algebraic numbers.

(i.) **The Multiplier Positive.** — Ex. **1.** Multiply $^+4$ by $^+3$.

By the definition of multiplication, the product,

$$^+4 \times {^+3},$$

is obtained from $^+4$ just as $^+3$ is obtained from the positive unit. But

$$^+3 = {^+1} + {^+1} + {^+1}.$$

Consequently the required product is obtained by taking $^+4$ *three times as a summand*, or

$$^+4 \times {^+3} = {^+4} + {^+4} + {^+4} = {^+(4 + 4 + 4)} = {^+(4 \times 3)} = {^+12}.$$

Ex. 2. Multiply ${}^-4$ by ${}^+3$.

By the definition of multiplication, we have

$$ {}^-4 \times {}^+3 = {}^-4 + {}^-4 + {}^-4 = {}^-(4 + 4 + 4) = {}^-(4 \times 3) = {}^-12. $$

(ii.) **The Multiplier Negative.** — Ex. 3. Multiply ${}^+4$ by ${}^-3$.

By the definition of multiplication, the product,

$$ {}^+4 \times {}^-3, $$

is obtained from ${}^+4$ just as ${}^-3$ is obtained from the positive unit. But

$$ {}^-3 = {}^-1 + {}^-1 + {}^-1 = - {}^+1 - {}^+1 - {}^+1 ; $$

that is, ${}^-3$ is obtained *by subtracting the positive unit,* ${}^+1$, *three times in succession from* 0. Consequently, the required product is obtained *by subtracting the multiplicand,* ${}^+4$, *three times in succession from* 0; or,

$$ {}^+4 \times {}^-3 = - {}^+4 - {}^+4 - {}^+4 = + {}^-4 + {}^-4 + {}^-4 = {}^-(4 \times 3). $$

Ex. 3. Multiply ${}^-4$ by ${}^-3$.

By the definition of multiplication, we have

$$ {}^-4 \times {}^-3 = - {}^-4 - {}^-4 - {}^-4 = + {}^+4 + {}^+4 + {}^+4 = {}^+(4 \times 3). $$

13. These examples illustrate the following **Rule of Signs for Multiplication:**

The product of two numbers having like signs is positive; and the product of two numbers having unlike signs is negative. Or, stated symbolically,

$$ {}^+\boldsymbol{a} \times {}^+\boldsymbol{b} = {}^+(\boldsymbol{ab}), \qquad {}^-\boldsymbol{a} \times {}^+\boldsymbol{b} = {}^-(\boldsymbol{ab}), $$

$$ {}^-\boldsymbol{a} \times {}^-\boldsymbol{b} = {}^+(\boldsymbol{ab}), \qquad {}^+\boldsymbol{a} \times {}^-\boldsymbol{b} = {}^-(\boldsymbol{ab}). $$

EXERCISES III.

Multiply:

1.	**2.**	**3.**	**4.**	**5.**	**6.**
${}^+3$	${}^-3$	${}^+3$	${}^-3$	${}^+8$	${}^-7$
${}^+4$	${}^+4$	${}^-4$	${}^-4$	${}^+5$	${}^-6$
7.	**8.**	**9.**	**10.**	**11.**	**12.**
${}^-9$	${}^+8$	${}^-12$	${}^-15$	${}^+20$	${}^+16$
${}^+2$	${}^-6$	${}^-5$	${}^+4$	${}^+7$	${}^-5$

DIVISION OF ALGEBRAIC NUMBERS.

14. Division *is the inverse of multiplication.* In multiplication two factors are given, and it is required to find their product. In division the product of two factors and one of them are given, and it is required to find the other factor.

E.g., Since $^{-}28 = {}^{-}4 \times {}^{+}7$,

therefore, $^{-}28 \div {}^{+}7 = {}^{-}4$, and $^{-}28 \div {}^{-}4 = {}^{+}7$.

15. From the definition of division we infer the following principle:

If the product of two numbers be divided by either of them, the quotient is the other number.

16. Since $^{+}\boldsymbol{a} \times {}^{+}\boldsymbol{b} = {}^{+}(\boldsymbol{ab})$, therefore $^{+}(\boldsymbol{ab}) \div {}^{+}\boldsymbol{a} = {}^{+}\boldsymbol{b}$;

since $^{-}\boldsymbol{a} \times {}^{+}\boldsymbol{b} = {}^{-}(\boldsymbol{ab})$, therefore $^{-}(\boldsymbol{ab}) \div {}^{-}\boldsymbol{a} = {}^{+}\boldsymbol{b}$;

since $^{-}\boldsymbol{a} \times {}^{-}\boldsymbol{b} = {}^{+}(\boldsymbol{ab})$, therefore $^{+}(\boldsymbol{ab}) \div {}^{-}\boldsymbol{a} = {}^{-}\boldsymbol{b}$;

since $^{+}\boldsymbol{a} \times {}^{-}\boldsymbol{b} = {}^{-}(\boldsymbol{ab})$, therefore $^{-}(\boldsymbol{ab}) \div {}^{+}\boldsymbol{a} = {}^{-}\boldsymbol{b}$.

From these equations we derive the following **Rule of Signs for Division:**

Like signs of dividend and divisor give a positive quotient; unlike signs of dividend and divisor give a negative quotient.

E.g., $^{+}8 \div {}^{+}2 = {}^{+}4$; $^{-}8 \div {}^{-}2 = {}^{+}4$;

$^{+}8 \div {}^{-}2 = {}^{-}4$; $^{-}8 \div {}^{+}2 = {}^{-}4$.

EXERCISES IV.

Divide:

1.	2.	3.	4.	5.
$^{+}4)\underline{^{+}8}$	$^{+}4)\underline{^{-}8}$	$^{-}4)\underline{^{+}8}$	$^{-}4)\underline{^{-}8}$	$^{-}5)\underline{^{-}15}$
6.	**7.**	**8.**	**9.**	**10.**
$^{-}7)\underline{^{+}28}$	$^{+}6)\underline{^{-}30}$	$^{+}8)\underline{^{+}24}$	$^{-}9)\underline{^{+}18}$	$^{+}3)\underline{^{-}27}$

ONE SET OF SIGNS FOR QUALITY AND OPERATION.

17. Most text-books of Algebra use the one set of signs, $+$ and $-$, to denote both *quality* and *operation*. We shall in subsequent work follow this custom. For the sake of brevity, the sign $+$ is usually omitted when it denotes *quality;* the sign $-$ is never omitted.

Thus, instead of $^{+}2$, we shall write $+2$, or 2;

instead of $^{-}2$, we shall write -2.

18. We have used the double set of signs hitherto in order to emphasize the difference between *quality* and *operation*. It should be kept clearly in mind that the same distinction still exists.

We now have

$^{+}3 + {}^{+}2 = +3 + (+2) = 3 + 2$, omitting the signs of *quality*, $+$;

$^{+}3 + {}^{-}2 = +3 + (-2)$, wherein $+$ denotes *operation*, and $-$ denotes *quality*.

$^{+}3 - {}^{+}2 = +3 - (+2) = 3 - 2$, omitting the signs of *quality*, $+$;

$^{+}3 - {}^{-}2 = +3 - (-2)$, wherein the first sign, $-$, denotes *operation*, the second sign, $-$, denotes *quality*.

19. In the chain of operations

$$(+2) + (-5) - (+2) - (-11)$$

the signs within the parentheses denote *quality*, those without denote *operation*. That expression reduces to

$$(+2) - (+5) - (+2) + (+11),$$

or

$$2 - 5 - 2 + 11,$$

dropping the sign of *quality*, $+$.

In the latter expression all the signs denote *operation*, and the numbers are all *positive*.

20. The following examples illustrate the double use of the signs $+$ and $-$.

Ex. 1. $^+4 + {}^+3 = +4 + (+3) = 4 + 3 = 7.$

Ex. 2. $^-5 + {}^+2 = -5 + (+2) = -5 + 2 = -3.$

Ex. 3. $^+7 - {}^-5 = +7 - (-5) = 7 - (-5) = 7 + 5 = 12.$

Ex. 4. $^-4 \times {}^+3 = -4 \times (+3) = -4 \times 3 = -12.$

Ex. 5. $^-4 \times {}^-3 = -4 \times (-3) = 12.$

Continued Products.

21. The results of Article 13 may be applied to determine the value of a chain of indicated multiplications, *i.e.*, of a *continued product*.

E.g.
$$(+a)(+b)(+c) = (+ab)(+c) = +abc,$$
$$(+a)(+b)(-c) = (+ab)(-c) = -abc,$$
$$(+a)(-b)(-c) = (-ab)(-c) = +abc,$$
$$(-a)(-b)(-c) = (+ab)(-c) = -abc.$$

These equations illustrate a more general rule of signs:

A continued product which contains no negative factor, or an even number of negative factors, is positive; one that contains an odd number of negative factors is negative.

In practice the sign of a required product may first be determined by inspection, and that sign prefixed to the product of the absolute values of the factors in the continued product.

E.g., the sign of the product

$$2 \times (-3) \times (-7) \times (+4) \times (-5)$$

is *negative*, since it contains *three* negative factors; the product of the absolute values is 840. Consequently,

$$2 \times (-3) \times (-7) \times (+4) \times (-5) = -840.$$

EXERCISES V.

In the expressions in Exx. 1–4, which signs denote quality and which operation?

1. $+5+(-3)-(+8)$. **2.** $-7+(+5)-(-9)$.

3. $-3+(-5)\times(+4)$. **4.** $(+12)\div(-4)\times(-3)$.

5–8. Find the value of the expressions in Exx. 1–4.

Find the values of the expressions in Exx. 9–20, first changing them into equivalent expressions in which there is only one set of signs $+$ and $-$:

9. ${}^{+}8+{}^{+}2$. **10.** ${}^{+}7-{}^{+}3$. **11.** ${}^{+}3-{}^{+}7$. **12.** ${}^{-}5+{}^{-}7$.

13. ${}^{-}8-{}^{+}3$. **14.** ${}^{-}9-{}^{-}5$. **15.** ${}^{+}4\times{}^{+}5$. **16.** ${}^{+}5\times{}^{-}2$.

17. ${}^{-}5\times{}^{-}2$. **18.** ${}^{+}12\div{}^{+}3$. **19.** ${}^{+}12\div{}^{-}3$. **20.** ${}^{-}12\div{}^{-}3$.

Simplify the following expressions:

21. $10-4$. **22.** $4-10$. **23.** $-8-7$.

24. $9-2$. **25.** $2-9$. **26.** $-10+10$.

27. 8×5. **28.** -8×5. **29.** $8\times(-5)$.

30. $(-8)\times(-5)$. **31.** $20\div 4$. **32.** $-20\div 4$.

33. $20\div(-4)$. **34.** $(-20)\div(-4)$. **35.** $-45\div 9$.

36. $3\times 5+4\times 2$. **37.** $3\times(5+4\times 2)$.

38. $8\times 6-10\div 5$. **39.** $(8\times 6-10)\div 5$.

40. $12\div 4-10\div 2$. **41.** $12\div(4-10\div 2)$.

When $a=16,\ b=-8,\ c=-2,\ d=-4$, find the values of:

42. $a+b+c$. **43.** $a+b-c$. **44.** $a-b+c$.

45. $a-b-c$. **46.** $a-(b-c)$. **47.** $c-(b-a)$.

48. abc. **49.** $ab\div c$. **50.** $a\div(bc)$.

51. $a\div b\times c$. **52.** $abcd$. **53.** $(ab)\div(cd)$.

54. $abc\div d$. **55.** $ab+cd$. **56.** $a\div b-d\div c$.

57. A's assets are \$2600 and B's are \$2200. How much do A's assets exceed B's, taking assets positively?

58. A owes \$200, and B's assets are \$1800. How much do A's assets exceed B's, taking assets positively?

59. The temperature in a room is 72° above zero, and out of doors it is 8° above zero. How much higher is the temperature in the room than out of doors, taking degrees above zero positively?

60. The temperature in a room is 70° above zero, and out of doors it is 4° below zero. How much higher is the temperature in the room than out of doors, taking degrees above zero positively?

PARENTHESES.

22. The **Terms** of an algebraic sum are the *additive* and *subtractive* parts of the sum.

E.g., the terms of $2 - 5 - 2 + 11$ are $+2, -5, -2, +11$

The **Sign of a Term** is its sign $+$ or $-$.

A **Positive Term** is one whose sign is $+$; as $+2$.

A **Negative Term** is one whose sign is $-$; as -5.

Removal of Parentheses.

23. We have $\quad 9 + (5 + 6) = 9 + 5 + 6,$

since to add the sum $5 + 6$ is equivalent to adding successively the single numbers of that sum.

Again, $\quad 9 + (5 - 6) = 9 + [5 + (-6)],$

since to *add* -6 is equivalent to *subtracting* 6.

Therefore, removing brackets,

$$9 + (5 - 6) = 9 + 5 + (-6), \ = 9 + 5 - 6.$$

The above example illustrates the following principle:

(i.) *When the sign of addition, $+$, precedes parentheses, they may be removed, and the signs, $+$ and $-$, within them be left unchanged;* that is,

$$\boldsymbol{N} + (+\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{N} + \boldsymbol{a} + \boldsymbol{b},$$

$$\boldsymbol{N} + (+\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{N} + \boldsymbol{a} - \boldsymbol{b}, \text{ etc.}$$

It is important to notice that if the first term within the parentheses has no sign, the sign $+$ is understood.

24. We also have

$$9-(5+6)=9-5-6,$$

since to subtract the sum $5+6$ is equivalent to subtracting successively the single numbers of that sum.

Again, $\quad 9-(5-6)=9-[5+(-6)],$

since to *add* -6 is equivalent to *subtracting* 6.

Therefore, removing brackets,

$$9-(5-6)=9-5-(-6), \;=9-5+6.$$

This example illustrates the following principle:

(ii.) *When the sign of subtraction, $-$, precedes parentheses, they may be removed, if the signs within them be reversed from $+$ to $-$, and from $-$ to $+$*; that is,

$$\boldsymbol{N}-(+\boldsymbol{a}+\boldsymbol{b})=\boldsymbol{N}-\boldsymbol{a}-\boldsymbol{b},$$
$$\boldsymbol{N}-(+\boldsymbol{a}-\boldsymbol{b})=\boldsymbol{N}-\boldsymbol{a}+\boldsymbol{b}, \text{ etc.}$$

Observe that the sign before the parentheses affects each term within them.

Insertion of Parentheses.

25. The insertion of parentheses is the converse of the process of removing them.

(i.) *An expression may be inclosed within parentheses preceded by the sign $+$, if the signs of the terms inclosed remain unchanged.*

E.g., $\quad 7-5+3-4=7+(-5+3-4),$

$$=7-5+(3-4).$$

(ii.) *An expression may be inclosed within parentheses preceded by the sign $-$, if the signs of the terms inclosed be reversed, from $+$ to $-$ and from $-$ to $+$.*

E.g., $\quad 7-5+3-4=7-(5-3+4),$

$$=7-5-(-3+4).$$

EXERCISES VI.

Find the value of each of the following expressions, first removing parentheses:

1. $9+(4+3)$.
2. $9+(4-3)$.
3. $10-(3+4)$.
4. $10-(3-4)$.
5. $12+(6+8)$.
6. $12-(6+8)$.
7. $12-(6-8)$.
8. $12+(-6+8)$.
9. $12-(-6+8)$.
10. $15+(9-6+2)$.
11. $15-(9-6+2)$.
12. $20-(7-9-1)$.
13. $18+(-4+5-8)$.
14. $18-(-4+5-8)$.

Insert parentheses in $10-7+4-6$ and $7+8-9-4$,

15. To inclose the last two terms, preceded by the sign $+$; preceded by the sign $-$.

16. To inclose the last three terms, preceded by the sign $+$; preceded by the sign $-$.

The Associative Law.

26. The principle for inserting parentheses enables us to group successive terms in algebraic addition.

E.g., $8+(4+1)=(8+4)+1$, or $8+5=12+1$.

In general, $a+(b+c)=(a+b)+c$.

That is, *the algebraic sum of three or more numbers is the same in whatever way successive numbers are grouped or associated in the process of adding.*

This principle is called the **Associative Law** for addition and subtraction.

27. In finding the value of a continued product in Art. 21, the indicated operations were performed successively from left to right.

E.g., $4\times 3\times(-2)=12\times(-2)=-24$.

But the result will be the same if 3 be first multiplied by -2, and then 4 be multiplied by this product.

E.g., $4 \times [3 \times (-2)] = 4 \times (-6) = -24.$

In like manner,

$$32 \times 4 \div 2 = 128 \div 2 = 64, \text{ and } 32 \times (4 \div 2) = 32 \times 2 = 64;$$
$$32 \div 4 \times 2 = \quad 8 \times 2 = 16, \text{ and } 32 \div (4 \div 2) = 32 \div 2 = 16;$$
$$32 \div 4 \div 2 = \quad 8 \div 2 = \ 4, \text{ and } 32 \div (4 \times 2) = 32 \div 8 = 4.$$

In general, $(\boldsymbol{ab})\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{bc})$; $\boldsymbol{ab} \div \boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b} \div \boldsymbol{c})$;

$$\boldsymbol{a} \div \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a} \div (\boldsymbol{b} \div \boldsymbol{c}); \quad \boldsymbol{a} \div \boldsymbol{b} \div \boldsymbol{c} = \boldsymbol{a} \div (\boldsymbol{bc}).$$

That is, *if a chain of multiplications and divisions be inclosed in parentheses, the symbols* $\times$ *and* $\div$, *preceding the numbers inclosed,*

(i.) *are unchanged if the symbol,* $\times$, *precede the parentheses;*

(ii.) *are reversed, from* $\times$ *to* $\div$ *and from* $\div$ *to* $\times$, *if the symbol,* $\div$, *precede the parentheses.*

This principle is called the **Associative Law** for multiplication and division.

The Commutative Law.

28. In an indicated addition, the number on the right of the symbol is to be added to the number on its left.

E.g., in $5 + 3, = 8$, 3 is added to 8, while in $3 + 5, = 8$, 5 is added to 3. But the results are the same.

That is, $5 + 3 = 3 + 5.$

In like manner, $8 - 5 = -5 + 8.$

In general, $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c} = \boldsymbol{a} - \boldsymbol{c} + \boldsymbol{b} = \text{etc.}$

That is, *the algebraic sum of two or more numbers is the same in whatever order they may be added.*

This principle is called the **Commutative Law** for addition and subtraction.

29. We have

$$4 \times 3 \times 2 = 12 \times 2 = 24, \text{ and } \ 4 \times 2 \times 3 = \ 8 \times 3 = 24;$$
$$14 \div 2 \times 7 = \ 7 \times 7 = 49, \text{ and } 14 \times 7 \div 2 = 98 \div 2 = 49;$$
$$8 \div 4 \div 2 = \ 2 \div 2 = \ 1, \text{ and } \ 8 \div 2 \div 4 = \ 4 \div 4 = 1.$$

In general,

$$a \times b \times c = a \times c \times b;\quad a \div b \times c = a \times c \div b;\quad a \div b \div c = a \div c \div b.$$

That is, *the result of a chain of multiplications and divisions is the same in whatever order these operations are performed.*

This principle is called the **Commutative Law** for multiplication and division.

30. By the preceding articles we have:

$$8 - 3 + 2 - 5 = 8 + 2 - 3 - 5 = 10 - 8 = 2 \qquad \text{(i.)}$$

$$25 \times 27 \times 4 = 25 \times 4 \times 27 = 100 \times 27 = 2700, \qquad \text{(ii.)}$$

$$75 \times 29 \div 25 = 75 \div 25 \times 29 = 3 \times 29 = 87. \qquad \text{(iii.)}$$

In changing the order of the operations, it is important to *carry the symbol of operation with the number.*

31. Thus, by the methods of the preceding article, we secure the following advantages:

In a succession of additions and subtractions, add the positive terms separately, then the negative terms, and unite the results.

In a succession of multiplications and divisions, we may, by changing the order of the operations, often simplify the work.

EXERCISES VII.

Find the value of each of the following expressions:

1. $8 - 3 + 2 - 5 + 9.$ **2.** $-6 + 4 - 14 + 12 - 7.$

3. $19 - 7 + 3 - 5 - 10.$ **4.** $16 - 7 + 4 - 9 + 3.$

5. $17 + 2 - 3 + 9 - 18.$ **6.** $15 - 19 + 6 - 7 + 5.$

Find, in the most convenient way, the value of each of the following expressions:

7. $89 - 115 + 11.$ **8.** $45\frac{2}{5} - 85 + 54\frac{3}{5}.$

9. $996 + 1008 + 4 - 8.$ **10.** $98 + 96 + 92 + 2 + 4 + 8.$

11. $25 \times 32 \times (-4).$ **12.** $12\frac{1}{2} \times (-29) \times 8.$

13. $-39 \times 16\frac{2}{3} \times 6.$ **14.** $45 \times 28 \div 9.$

15. $-12\frac{1}{2} \div 20 \times 8.$ **16.** $10 \div 42 \times 21.$

POSITIVE INTEGRAL POWERS.

32. The **Sign of Continuation,** $\cdots$, is read, *and so on*, or *and so on to;* as 1, 2, 3, $\cdots$, read, *one, two, three, and so on;* or 1, 2, 3, $\cdots$, 10, read, *one, two, three, and so on to* 10.

33. A continued product of equal factors is called a **Power** of that factor.

Thus, 2×2 is called the *second power* of 2, or 2 *raised to the second power;* aaa is called the *third power* of a, or a *raised to the third power.*

In general $aaa \cdots$ to n factors is called the nth power of a, or a *raised to the nth power.*

The second power of a is often called the *square* of a, or a *squared;* and the third power of a the *cube* of a, or a *cubed.*

34. The notation for powers is abbreviated as follows:

a^2 is written instead of aa; a^3 instead of aaa;

a^n instead of $aaa \cdots$ to n factors.

35. The **Base** of a power is the number which is repeated as a factor.

E.g., a is the base of $a^2, a^3, \cdots, a^n$.

36. The **Exponent** of a power is the number which indicates how many times the base is used as a factor, and is written to the right and a little above the base.

E.g., the exponent of a^2 is 2, of a^3 is 3, of a^n is n.

The exponent 1 is usually omitted. Thus, $a^1 = a$.

37. The base of a power must be inclosed within parentheses to prevent ambiguity:

(i.) *When the base is a negative number.* Thus,

$$(-5)^2 = (-5)(-5) = 25; \text{ while } -5^2 = -(5 \times 5) = -25.$$

(ii.) *When the base is a product or a quotient.* Thus,

$$(2 \times 5)^3 = (2 \times 5)(2 \times 5)(2 \times 5) = 1000;$$

while $$2 \times 5^3 = 2(5 \times 5 \times 5) = 250.$$

Likewise $\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, while $\frac{2^2}{3} = \frac{2 \times 2}{3} = \frac{4}{3}$.

(iii.) *When the base is a sum.* Thus,

$$(2+3)^2 = (2+3)(2+3) = 5 \times 5 = 25;$$

while $$2 + 3^2 = 2 + 3 \times 3 = 2 + 9 = 11.$$

(iv.) *When the base is itself a power.* Thus,

$$(2^3)^2 = 2^3 \times 2^3 = (2 \times 2 \times 2)(2 \times 2 \times 2) = 64.$$

while $2^{3^2} = 2^{3 \times 3} = 2^9 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 512.$

EXERCISES VIII.

Express each of the following powers in the abbreviated notation:

1. $a \times a$. **2.** 4×4. **3.** $2 \times 2 \times 2$. **4.** $(-a)(-a)$.

5. $-a \times a$. **6.** $(-3)(-3)(-3)(-3)$. **7.** $-nnnnn$.

8. $2 \times 2 \times 2 \cdots$ to 8 factors.

9. $(-a)(-a)(-a) \cdots$ to 9 factors.

10. $(a+b)(a+b)(a+b)$. **11.** $(x-yy)(x-yy)(x-yy)$.

12. $(a+b)(a+b)(a+b) \cdots$ to 12 factors.

Express each of the following powers as a continued product:

13. 3^6. **14.** 6^3. **15.** -4^3. **16.** $(-4)^3$.

17. xy^3. **18.** $(xy)^3$. **19.** $(-a)^4$. **20.** $-a^4$.

Write:

21. Four times x. **22.** x to the fourth power.

23. The sum of the cubes of a and b.

24. The cube of the sum of a and b.

25. The length of a side of a square floor is a feet. How many square feet in the floor?

26. A field is $3a$ rods long and $2a$ rods wide. How many square rods in its area?

27. A box is $4x$ feet long, $3x$ feet wide, and $2x$ feet high. How many cubic feet does it contain?

Properties of Positive Integral Powers.

38. (i.) *All (even and odd) powers of positive bases are positive.*

E.g., $2^3 = 2 \times 2 \times 2 = 8.$ $\quad 3^4 = 3 \times 3 \times 3 \times 3 = 81.$

(ii.) *Even powers of negative bases are positive; odd powers of negative bases are negative.*

E.g., $$(-2)^4 = (-2)(-2)(-2)(-2) = 16;$$
$$(-5)^3 = (-5)(-5)(-5) = -125.$$

In general, $$(+a)^m = +a^m;$$
$$(-a)^{2n} = a^{2n}; \quad (-a)^{2n+1} = -a^{2n+1}.$$

EXERCISES IX.

Find the value of each of the following powers:

1. 2^5. **2.** 5^2. **3.** $(-2)^6$. **4.** -2^6. **5.** $(-3)^5$.
6. $(-2)^8$. **7.** -3^3. **8.** $(-3)^3$. **9.** $(-a)^6$. **10.** $(-a)^9$.

Express as powers of 2:

11. 8. **12.** 32. **13.** 128. **14.** 1024. **15.** 4096.

Express as powers of -3:

16. 9. **17.** -27. **18.** -243. **19.** 729. **20.** -2187.

Find the value of each of the following expressions:

21. $2^2 + 3^2$. **22.** $(2+3)^2$. **23.** $3^3 - 2^3$. **24.** $(3-2)^3$.
25. $(4 \times 3)^2$. **26.** 6×4^2. **27.** $2(-3)^3$. **28.** $[2(-3)]^3$.

When $a = 5$, $b = -4$, $c = 2$, find the value of each of the following expressions:

29. a^c. **30.** b^a. **31.** $(ab)^c$. **32.** bc^a. **33.** $(abc)^c$.
34 $(a-b-c)^2$. **35.** $a^2 - b^2 - c^2$. **36.** $(a^2 - b^2 + c^2)^2$.

CHAPTER III.

THE FUNDAMENTAL OPERATIONS WITH INTEGRAL ALGEBRAIC EXPRESSIONS.

DEFINITIONS.

1. An **Integral Algebraic Expression** is an expression in which the *literal* numbers are connected only by one or more of the symbols of operation, $+$, $-$, $\times$, but not by the symbol $\div$.

E.g., $1 + x + x^2$, $5a^2b + \frac{3}{8}cd^2$, etc.

2. The word *integral* refers only to the *literal* parts of the expression.

E.g., $a + b$ is *algebraically* integral; but when $a = \frac{1}{2}$, $b = \frac{3}{4}$, we have

$$a + b = \tfrac{1}{2} + \tfrac{3}{4} = 1\tfrac{1}{4}.$$

3. Coefficients. — In a product, any factor, or product of factors, is called the **Coefficient** of the product of the remaining factors.

E.g., in $3abc$, 3 is the coefficient of abc, $3b$ of ac, etc.

A **Numerical Coefficient** is a coefficient expressed in figures.

E.g., in $-3ab$, -3 is the numerical coefficient of ab.

A **Literal Coefficient** is a coefficient expressed in letters, or in letters and figures.

E.g., in $3ab$, a is the literal coefficient of $3b$, and $3a$ of b.

The coefficients $+1$ and -1 are usually omitted.

4. A coefficient must not be confused with an exponent.

E.g., $4a = a + a + a + a$; while $a^4 = a \times a \times a \times a$.

5. The sign $+$, or the sign $-$, preceding a product, is to be regarded as the sign of its numerical coefficient.

Thus $+3a$ means the product of *positive* 3 by a; $-5x$ means the product of *negative* 5 by x. In particular, $+a$ means the product of *positive* 1 by a, and $-a$ means the product of *negative* 1 by a, unless the contrary is stated.

EXERCISES I.

What is the coefficient of x in

1. $2x$? 2. $-3x$? 3. $5ax$? 4. $-7bx$?

5. If the sum, $a + a + a + a$, be represented as a product, what is the coefficient of a?

6. If the algebraic sum, $-b-b-b-b-b$, be represented as a product, what is the coefficient of $-b$? Of b?

7. If the sum $ax + ax + ax + \cdots$ to 10 terms be represented as a product, what is the coefficient of ax? Of x?

6. **Like** or **Similar Terms** are terms which do not differ, or which differ only in their numerical coefficients.

E.g., in the expression $+3a+6ab-5a+7ab$, $+3a$ and $-5a$ are like terms; so are $+6ab$ and $+7ab$.

Unlike or **Dissimilar Terms** are terms which are not like.

E.g., $+3a$ and $+7ab$ in the above expression.

7. A **Monomial** is an expression of one term; as a, $-7bc$.

A **Binomial** is an expression of two terms; as $-2a^2+3bc$.

A **Trinomial** is an expression of three terms.

E.g., $a+b-c$, $-3a^2+7b^3-5c^4$.

A **Multinomial*** is an expression of two or more terms, including, therefore, binomials and trinomials as particular cases.

E.g., $a+b^2$, a^2+b-c^3, $ab+bc-cd-ef$.

* The word **Polynomial** is frequently used instead of **Multinomial**.

ADDITION AND SUBTRACTION.

Addition of Like Terms.

8. Like Terms can be united by addition into a single *like* term.

Just as $2 = 1 + 1$, so $2\,xy = xy + xy$;

just as $3 = 1 + 1 + 1$, so $3\,xy = xy + xy + xy$.

Therefore, just as $2 + 3 = 5$,

so $$2\,xy + 3\,xy = (2 + 3)\,xy = 5\,xy.$$

That is, *to add like terms, add their numerical coefficients and annex to the sum their common literal part.*

Ex. 1. Add $-7\,ab$ to $4\,ab$.

We have $4\,ab + (-7\,ab) = [4 + (-7)]\,ab = -3\,ab.$

Ex. 2. Find the sum of $3\,a,\ -5\,a,\ 8\,a,\ -4\,a.$

Uniting the positive terms by themselves, and the negative terms by themselves, we have

$$3\,a + 8\,a + (-5\,a) + (-4\,a) = [3 + 8 + (-5) + (-4)]\,a = 2\,a.$$

Ex. 3. Add ax to bx.

Since the sum of the coefficients of x is $a + b$, we have

$$ax + bx = (a + b)\,x.$$

EXERCISES II.

Add:

1.	2.	3.	4.	5.	6.
a	$-3\,b$	$2\,x$	$-3\,m$	$7\,a$	$-5\,x$
$2\,a$	$-5\,b$	$7\,x$	$-15\,m$	$12\,a$	$-4\,x$
$3\,a$	$-2\,b$	$5\,x$	$-11\,m$	$-5\,a$	$3\,x$

7.	8.	9.	10.	11.	12.
$-9\,a^2$	$4\,xy$	$-7\,x^2y$	ax	ay	$-mx^2$
$11\,a^2$	$-15\,xy$	$3\,x^2y$	bx	$-by$	nx^2
$-5\,a^2$	$12\,xy$	$-6\,x^2y$	cx	cy	$-px^2$

Find the sum of:

13. $4a,\ 5a,\ 7a,\ 9a.$ **14.** $-5x,\ -3x,\ -9x,\ -13x.$

15. $6a^2,\ -3a^2,\ 11a^2,\ -2a^2.$

16. $-11xy,\ 17xy,\ 5xy,\ -4xy.$

17. $8a^2b,\ -3a^2b,\ 27a^2b,\ -11a^2b,\ -21a^2b.$

18. $m+n,\ -5(m+n),\ 9(m+n),\ -4(m+n).$

19. $3(a^2+b),\ -8(a^2+b),\ -14(a^2+b),\ a^2+b.$

Simplify the following expressions:

20. $5x-13x+9x.$ **21.** $7a-9a-4a.$

22. $5m+13m-8m.$ **23.** $a^2-7a^2+5a^2.$

24. $-a^2b+15a^2b-8a^2b+14a^2b.$

25. $2a^3-15a^3+11a^3+12a^3-9a^3.$

26. $-7x^2y^2+13x^2y^2-8x^2y^2-3x^2y^2+5x^2y^2.$

27. $12a^3b-15a^3b-8a^3b+20a^3b-8a^3b.$

28. $a+b-3(a+b)+8(a+b)+5(a+b)-10(a+b).$

29. $5(x^2+y^2)+8(x^2+y^2)-11(x^2+y^2)+3(x^2+y^2).$

30. $x+\frac{1}{3}x-\frac{2}{3}x-\frac{1}{2}x.$ **31.** $3y+\frac{1}{4}y-\frac{1}{6}y-\frac{2}{3}y.$

32. $\frac{1}{2}a-\frac{1}{4}a+\frac{5}{6}a-\frac{7}{8}a+\frac{3}{2}a+\frac{5}{4}a-\frac{11}{6}a.$

Simplify the following expressions, first removing parentheses:

33. $2a-[-4a-(-6a)].$ **34.** $m+[2m-(3m-4m)].$

35. $6y-[5y-4y-(-3y+2y)]-y.$

36. $x-[x-2x-(x-3x)-(x-4x)].$

Subtraction of Like Terms.

9. Like Terms can be united by subtraction into a single like term.

Just as
$$5-2=3,$$
so
$$5a-2a=(5-2)a=3a.$$

That is, *to subtract like terms, subtract their numerical coefficients, and annex to the remainder their common literal part.*

Ex. 1. Subtract $-5x^2y$ from $-7x^2y$.

We have

$$-7x^2y-(-5x^2y)=-7x^2y+5x^2y=(-7+5)x^2y=-2x^2y.$$

EXERCISES III.

Subtract:

1.	2.	3.	4.	5.	6.
$5a$	$7x$	$-5m$	$-8y$	$6a$	$11x$
a	$3x$	$2m$	$4y$	$-3a$	$-5x$

7.	8.	9.	10.	11.	12.
$-3a$	$-11m$	$3a^2$	$7m^3$	a^2b	x^3y
$-5a$	$-12m$	$5a^2$	$8m^3$	$-3a^2b$	$-2x^3y$

13. $13a^2b$ from $15a^2b$.

14. $-7x^3y^3$ from $3x^3y^3$.

15. $\frac{2}{5}xy^2$ from $\frac{3}{2}xy^2$.

16. $-\frac{3}{4}ab^5$ from $\frac{5}{6}ab^5$.

17. $2(a+b)$ from $-3(a+b)$.

18. x^2+y^2 from $-2(x^2+y^2)$.

Addition of Multinomials.

10. Unlike Terms are added by writing them in succession each preceded by the sign $+$.

Ex. 1. Add $3b$ to $2a$. We have $2a+3b$.

Ex. 2. Add $-3x^2$ to $2y^2$. We have

$$2y^2+(-3x^2)=2y^2-3x^2.$$

Such steps as changing $+(-3x^2)$ into $-3x^2$, should be performed mentally.

11. A multinomial consisting of two or more sets of like terms can be simplified by uniting like terms.

Ex. 1.
$$\begin{aligned}2a-3b-5a+4b&=2a-5a-3b+4b\\&=-3a+b.\end{aligned}$$

12. If two or more multinomials have common like terms, these terms can be united.

Ex. 1. Add $-2a+3b$ to $3a-5b$.

We have $(3a-5b)+(-2a+3b)=3a-5b-2a+3b,$
$$=a-2b.$$

In adding multinomials, it is often convenient to write one underneath the other, placing like terms in the same column.

Ex. 2. Find the sum of $-4x^2+3y^2-8z^2$, $2x^2-3z^2$, and $2y^2+5z^2$.

We have
$$\begin{array}{r} -4x^2+3y^2-8z^2 \\ 2x^2 \qquad\quad -3z^2 \\ 2y^2+5z^2 \\ \hline -2x^2+5y^2-6z^2 \end{array}$$

It is evidently immaterial whether the addition is performed from left to right, or from right to left, since there is no carrying as in arithmetical addition.

EXERCISES IV.

Add:

1.	2.	3.	4.	5.	6.
a	2	3	-4	a	$-m$
1	b	x	c	b	$-n$

7. a to a^2. 8. $-x$ to x^2. 9. $-2m$ to n.

10. x^2 to $-2xy$. 11. xy to yz. 12. a^2b to ab^2.

Simplify the following expressions by uniting like terms:

13. $a+2+a-2$. 14. $5b-3-4b+4$.

15. $10x-8+5-7x$. 16. $9m-3n-8m+4n$.

17. $-a^3-5a^2+4a^2+2a^2+2a^3$.

18. $ab-3a^2b^2+5ab-8a^2b^2+4ab+a^2b^2$.

19. $-3(a^2+b)+5(a+b^2)+4(a^2+b)-4(a+b^2)$.

Simplify the following expressions, first removing parentheses:

20. $a+1-(2-3a)$. 21. $5x-(-2y+3x)$.

22. $x-2y-(2y+3x)-(3x-4y)$.

23. $2m + 3n - (5m - 4n) - (-3m + 7n)$.

24. $2xy + 5yz - (2xy - 3yz) + 2xy - (3xy - 2yz) + 5yz$.

25. $a - [3a - (2a + b)] - (3b - 5)$.

26. $3x - [x + 3y - (y - 2x)]$.

27. $8m - [m - (3m - n) + (2m - 3n)]$.

Find the values of the expressions in Exx. 20–24,

28. When $a = 1$, $x = 3$, $y = -5$, $z = 10$, $m = 4$, $n = -7$.

29. When $a = -3$, $x = 6$, $y = -7$, $z = 8$, $m = -1$, $n = -2$.

Find the sum of the following expressions:

30. $5a + 2b$, $3a - 4b$, $-7a + 3b$, $9a - b$.

31. $7x - 3y$, $5x + 4y$, $-10x + 4y$, $3x - 7y$.

32. $a + 2b - 3c$, $2a - 3b + c$, $2a + 5b + 2c$.

33. $a^2 + 2a + 1$, $a^2 - 3a - 2$, $a^2 + 4a + 2$.

34. $x^2 - 5x + 6$, $3x^2 + 2x - 7$, $6x^2 + 3x + 1$.

35. $2ab + 3ac - 4$, $3ab - 5ac + 2$, $-5ab + 3ac + 8$.

36. $a^2 - 3ab + b^2$, $2a^2 + 2ab - b^2$, $ab - 2a^2$.

37. $a^3 - 5a^2b$, $7a^2b - b^3$, $a^3 + b^3$.

38. $x^3 - 3x^2 + 5x - 1$, $7x^3 + 2x^2 - 6x + 4$, $-2x^3 - 3x^2 + 4x - 5$.

39. $x^3 + 5x^2y - 7xy^2 - 2y^3$, $-2x^3 + 6x^2y + 11xy^2 - 15y^3$, $4x^3 - 7x^2y - 5xy^2 + 3y^3$.

40. $a^4 + 2a^2 - 5a - 3$, $-3a^4 + 2a^3 + 6a - 4$, $2a^4 - 7a^3 + 3a^2 + 9$, $5a^4 - 7a^3 - 5a^2 + a$.

41. $2(x + y)^2 + 3(x + y)$, $-(x + y)^2 + (x + y)$, $-2(x + y) + 1$.

42. $a^2 - 2(a + b)^2 + b^2$, $a^2 + 3(a + b)^2$, $-a^2 - 2b^2$.

43. $\frac{1}{2}x + \frac{1}{5}y$, $-\frac{1}{3}x + \frac{1}{4}y$, $\frac{1}{6}x - \frac{1}{10}y$.

44. $\frac{2}{3}a^2b - \frac{1}{3}ab^2$, $-\frac{1}{2}a^2b + \frac{1}{4}ab^2$, $-\frac{5}{6}a^2b - \frac{7}{12}ab^2$.

45. $\frac{5}{6}x^2 - \frac{3}{4}xy + \frac{1}{10}y^2$, $\frac{1}{3}x^2 + \frac{2}{3}xy - \frac{1}{15}y^2$, $-\frac{5}{12}x^2 + \frac{7}{6}xy - \frac{3}{20}y^2$.

Subtraction of Multinomials.

13. **Unlike Terms** are subtracted by writing them in succession, each preceded by the sign $-$.

Ex. Subtract $-11m$ from $2n$. We have

$$2n-(-11m)=2n+11m.$$

14. If two multinomials have common like terms, these terms can be united.

Ex. 1. Subtract $-2a+3b$ from $3a-5b$.

We have $(3a-5b)-(-2a+3b)=3a-5b+2a-3b,$

$$=5a-8b.$$

Ex. 2. Subtract $2x^2-6x-3$ from $3x^2-5x+1$.

Changing mentally the signs of the terms of the subtrahend and adding, we have

$$\begin{array}{r} 3x^2-5x+1 \\ 2x^2-6x-3 \\ \hline x^2+\ \ x+4 \end{array}$$

Ex. 3. Subtract $2x^2-3z^2$ from $-4x^2+3y^2$, and from the result subtract $2y^2+5z^2$.

When several multinomials are to be subtracted in succession, the work is simplified by writing them with the signs of the terms already changed. We then have

$$\begin{array}{rrr} -4x^2 & +3y^2 & \\ -2x^2 & & +3z^2 \\ & -2y^2 & -5z^2 \\ \hline -6x^2 & +\ \ y^2 & -2z^2 \end{array}$$

EXERCISES V.

Subtract:

1.	2.	3.	4.	5.	6.
1	3	x	x^2	$-mn$	a^2b
$\underline{a}$	$\underline{-b}$	$\underline{y}$	$\underline{-x}$	$\underline{m}$	$\underline{-ab^2}$

7. $3a-2b$ from $4a-3b$. **8.** $-5x+4y$ from $-4x+5y$.

9. $7m+2n$ from $-3m+3n$.

10. $2x^2-3x$ from $3x^2-2x$.

11. $5a-7b+8c$ from $6a-6b+9c$.

12. $2x^2-5y^2+11z^2$ from $3x^2-7y^2+14z^2$.

13. $2xy+5xz-7yz$ from $5xy+3xz-6yz$.

14. $2a^2-3ab-12b^2$ from $3a^2-ab-11b^2$.

15. $7x^2y^2-5xy+8$ from $8x^2y^2-3xy+7$.

16. x^3-3x^2+5x-1 from $2x^3-2x^2+6x$.

17. $2a^3-a^2b-b^3$ from $3a^3+2a^2b+3ab^2$.

18. x^2-x-1 from x^3+2x^2.

19. $2(x+y)-5z$ from $3(x+y)-4z$.

20. $6(a-b)-3a+b^2$ from $5(a-b)-2a+a^2$.

21. From the sum of $5x-5y+3z$ and $4x+4y-2z$ subtract $8x-2y-2z$.

22. From a^2-ab+b^2 subtract the sum of $2a^2-3ab+5b^2$ and $a^2+ab-4b^2$.

23. How much does m^2+n^2 exceed m^2-n^2?

24. How much does $1-x^2$ exceed $2-3x^2$?

25. What expression must be added to $2a-3b+4c$ to give $4a+2b-2c$?

26. What expression must be added to $xy+xz+yz$ to give $x^2+y^2+z^2$?

27. What expression must be subtracted from a^2+ab+b^2 to give $a^2-2ab+b^2$?

28. What expression must be subtracted from $x^2-2xy+y^2$ to give $x^2+2xy+y^2$?

29. What expression must be added to x^2+x+1 to give 0?

If $x=2a-3b+4c$, $y=-3a+2b-7c$, $z=9a-7b+6c$, find the values of

30. $x+y+z$. **31.** $x-y+z$. **32.** $x+y-z$. **33.** $x-y-z$.

If $A=\frac{1}{2}x-\frac{2}{3}y+\frac{5}{6}z$, $B=-\frac{1}{4}x+\frac{1}{5}y-\frac{3}{8}z$, $C=-\frac{7}{8}x-\frac{5}{2}y+\frac{3}{4}z$, find the values of

34. $A+B+C$. **35.** $A-B+C$.

36. $A+B-C$. **37.** $A-B-C$.

PARENTHESES.

15. The use of parentheses has been briefly discussed in Ch. II., Arts. 23–25. It is frequently necessary to employ more than two sets of parentheses, and to distinguish them the following forms are used:

Parentheses, ( ); **Brackets,** [ ]; **Braces,** { }.

A **Vinculum** is a line drawn over an expression, and is equivalent to parentheses inclosing it.

E.g., $$(a+b)(c-d)=\overline{a+b}\cdot\overline{c-d}.$$

16. The principles given in Ch. II., Arts. 23–24, are to be applied successively when several sets of parentheses are to be removed from a given expression.

17. In removing parentheses we may begin either with the inmost or with the outmost.

The following example will illustrate the method of removing parentheses, beginning with the inmost:

Ex.
$$\begin{aligned}&4a-\{3a+[2a-(a-1)]\}\\&=4a-\{3a+[2a-a+1]\}\\&=4a-\{3a+a+1\}\\&=4a-4a-1=-1.\end{aligned}$$

When, in such examples, we come to one of a pair of parentheses, (, or [, or {, we must look for the other of like form. We then treat all that is contained in each pair as a whole.

EXERCISES VI.

Simplify each of the following expressions:

1. $2x-3y-[5x-(2y-3\overline{x-y})]$.
2. $a+2b-[6a-\{3b-(6a-6b)\}]$

3. $2x - \{3y - [4x - (5y - 6x)]\}$.

4. $6a - [7a - \{8a - (9a - \overline{10a - b})\}]$.

5. $a - \{5b - [a - (3c - 3b) + 2c - (a - 2b - c)]\}$.

6. $(7a - 6) - \{4a - [2a - 1 - (3 - \overline{4a - 5})]\}$.

7. $x - [x - \{2x - 3 - [4x - 5 - (6x - \overline{7x - 8})]\}]$.

8. $a - \{3a - [a - b + \{5a - b - (7a - 6 - \overline{8a - 6})\}]\}$.

9. Find the values of the expressions in Exx. 1–5, when $a = -3,\ b = 4,\ c = -5,\ x = 8,\ y = -9$.

18. Ex. 1. Express $4(x - y) + y - x$ as a product, of which one factor is $x - y$.

We have $4(x - y) + y - x = 4(x - y) - (x - y) = 3(x - y)$.

The sign + or − before a pair of parentheses can evidently be reversed from + to −, or from − to +, if the signs of the terms within the parentheses be reversed.

Ex. 2. $7(x - 1) - 3(1 - x) = 7(x - 1) + 3(x - 1) = 10(x - 1)$.

EXERCISES VII.

Write each of the following expressions as a product, of which the expression within the parentheses is one of the factors:

1. $3(a - b) - a + b$.
2. $5(x^2 - y) - x^2 + y$.
3. $3m - 5n - 4(5n - 3m)$.
4. $1 - a^n + 3(a^n - 1)$.
5. $5(x^2 - x + 1) - x^2 + x - 1$.
6. $x - y - z - 6(y + z - x)$.

Write each of the following expressions as a single product, of which the expression within the first parentheses is a factor:

7. $(2x - 1) - 3(1 - 2x)$.
8. $2(2m - 3n) + (3n - 2m)$.
9. $5(x^2 - y^2) + 2(y^2 - x^2)$.
10. $7(xy - z) - (z - xy)$.

Simplify the following expressions without removing the parentheses:

11. $(a - b)c + (b - a)c$.
12. $5(x - y)z + 5(y - x)z$.

EQUATIONS AND PROBLEMS.

19. Ex. Find the value of x from $2x - 5 = 7 + x$.

Adding 5 to both members of the equation, we obtain

$$2x - 5 + 5 = 7 + 5 + x;$$

or, since $-5 + 5 = 0$, $\quad 2x = 7 + 5 + x.$

Subtracting x from both members of the last equation, we have

$$2x - x = 7 + 5 + x - x;$$

or, since $x - x = 0$, $\quad 2x - x = 7 + 5. \qquad (1)$

Uniting terms, $\quad x = 12.$

Check: $2 \times 12 - 5 = 7 + 12$, or $24 - 5 = 7 + 12$, or $19 = 19$.

20. Observe that equation (1), Art. 19, could have been obtained directly from the given equation by transferring the term -5, with sign changed, to the second member, and the term $+x$, with sign changed, to the first member.

That is, *any term may be transferred from one member of an equation to the other, if its sign be reversed from* $+$ *to* $-$, *or from* $-$ *to* $+$.

21. Ex. Find the value of x from the equation $x - 3 = 8 - 3$.

Adding 3 to both members of the equation, we obtain:

$$x - 3 + 3 = 8 - 3 + 3;$$

or, since $-3 + 3 = 0$, $\quad x = 8.$

Check: $\quad 8 - 3 = 8 - 3$, or $5 = 5$.

Observe that this step is equivalent to dropping the common term -3 from both members.

That is, *the same term, or equal terms, may be dropped from both members of an equation.*

This step *is called cancellation of equal terms.*

22. These examples illustrate the following method:

Transfer all the terms containing the unknown number to one member of the equation, usually to the first member, and all the terms containing known numbers to the other member.

Unite like terms.

Divide both members by the coefficient of the unknown number.

Check *by substituting the value thus obtained in the given equation.*

23. Pr. A boy being asked his age, replied, "If 10 is added to twice the number of years in my age the sum will be 40." How old was the boy?

Let x stand for the number of years in his age.

Then $2x$ stands for twice that number of years.

The problem states,

in *verbal* language: *twice the number of years in the boy's age plus* 10 *is equal to* 40;

in *algebraic* language: $2x + 10 = 40.$

Transferring 10, $2x = 30.$

Dividing by 2, $x = 15.$

The boy was 15 years old.

Check: $2 \times 15 + 10 = 40$, or $30 + 10 = 40$, or $40 = 40$.

EXERCISES VIII.

Solve each of the following equations:

1. $x + 4 = 9.$ **2.** $3 + x = 10.$ **3.** $x - 5 = 6.$

4. $15 - x = 10.$ **5.** $11 - x = 13.$ **6.** $3x + 2 = 11.$

7. $5x - 3 = 17.$ **8.** $7 + 12x = 31.$ **9.** $41 - 17x = 7.$

10. $15 = 3 + 4x.$ **11.** $19 = 13 - 6x.$ **12.** $14 = 8 - 3x.$

13. $9 + 5x = 13 + 4x.$ **14.** $8x - 5 = 10x - 11.$

15. $18 - 5x = 33 - 8x.$ **16.** $14x - 13 = 7x + 29.$

17. $3x - 4 + 5x = 7x + 9.$ **18.** $4x - 9 = 8x - 3 - 2x.$

19. $2x+5x-33=8x-37-15.$

20. $11x-15-4x=2x+5-5x.$

21. $13x-25+7x=87+9x+9.$

22. $5x+14-8x=3x-16-4x.$

23. $6x+7-15x+23=36x+15.$

24. $6x-25+3x-14x=25-3x.$

25. $4x+(2x-3)=15.$

26. $2x-(5x+5)=7.$

27. $5x-(3x-7)=17.$

28. $7x-(3x-11)=4.$

29. $14x-\{3x-(2-x)\}=22.$

30. $3x-7-(5x+17)=0.$

31. $6x-[7x-(8x-18)]=16.$

32. $6-\{5-(4-\{3-[2-(1-x)]\})\}=4.$

33. If 19 is added to a number, the sum will be 40. What is the number?

34. A man invests \$2100. How much must he gain to have \$3600?

35. What number increased by 43 gives its double?

36. What number is 16 less than three times itself?

37. A pole 34 feet long is divided into two parts, so that one part is 8 feet shorter than the other. What is the length of each part?

38. In a number of 2 digits, the tens' digit exceeds the units' digit by 3. If the sum of the digits is 13, what is the units' digit? What is the number?

39. What is the number next greater than 8? Next less? Next greater than x? Next less? Next greater than $x+4$?

40. The sum of two consecutive numbers is 31. What are the numbers?

41. The sum of three consecutive numbers is 24. What are the numbers?

42. The difference between two numbers is 7, and the smaller number is 9. What is the greater number? If the greater number is x, what is the smaller number?

43. The difference between two numbers is 8, and their sum is 38. What are the numbers?

44. The difference between two numbers is 3, and their sum is equal to nine times their difference. What are the numbers?

45. A father is 40 years older than his son. If six times the son's age is equal to the sum of their ages, how old is each?

46. The length of a room is three times the breadth. If the length is 20 feet more than the breadth, what are the dimensions of the room?

47. A man, being asked the time, replied, "If 18 is subtracted from four times the hour it now is, the remainder will be the hour." What was the hour?

48. Three times a number exceeds 12 by as much as 12 exceeds the number. What is the number?

49. A has $125 and B has $45. How many dollars must A give B in order that they may have equal amounts?

50. A pile stands 3 feet above the water. If $\frac{1}{3}$ is in the water and $\frac{1}{6}$ in the earth, how long is the pile?

51. Two vessels together hold 9 gallons. If the smaller, when empty, is filled from the larger, when full, there will remain 3 gallons in the latter. How many gallons does each vessel hold?

52. Three boys, A, B, and C, pull 100 pounds. A pulls 20 pounds more than B, and B pulls 8 pounds less than C. How many pounds does each boy pull?

53. A pole is divided into three parts. The second is three times as long as the first, and the third is 6 feet longer than the first. The length of the pole is equal to the excess of 60 feet over the smallest part. What are the lengths of the parts, and the length of the pole?

54. In a number of two digits, the units' digit is three times the tens' digit. The number is equal to 8 more than three times the units' digit. What is the number?

MULTIPLICATION.

Product of Powers.

24. Ex. 1. $a^3 \times a^4 = (aaa)(aaaa) = aaaaaaa = a^7 = a^{3+4}$.

Ex. 2. $xx^2x^3 = x(xx)(xxx) = xxxxxx = x^6 = x^{1+2+3}$.

These examples illustrate the following principle:

The exponent of the product of two or more powers of the same base is the sum of the given exponents; or stated symbolically,

$$a^m a^n = a^{m+n};\ a^m a^n a^p = a^{m+n+p};\ \text{etc.}$$

EXERCISES IX.

Express each of the following products as a single power:

1. $3^2 \times 3$. **2.** $5^3 \times 5^2$. **3.** $6^4 \times 6^3$. **4.** $(-5)^4 5^2$.

5. $(-6)^3(-6)^4$. **6.** $2^5(-2)^7$. **7.** $8^3(-8)^4$. **8.** $(-7)^5 7^3$.

9. $x^5 \times x^6$. **10.** $(-y)^6 y^3$. **11.** $(-a)^3(-a)^4$. **12.** $(-x)^3 x^3$.

13. $a^3a^5a^7$. **14.** $x^4(-x)^6x^2$. **15.** $a^2a^4a^5a^3$.

16. $(xy)^3(xy)^4$. **17.** $(2ab)^3[-(2ab)]^4$.

18. $(a+b)^3(a+b)^5$. **19.** $[-(x-y)]^3(x-y)^5$.

20. $x^n x^{3n}$. **21.** $a^n a^2$. **22.** $x^{n-1}x$. **23.** $y^n y^{2-n}$.

24. $z^{n+1}z^{n-1}$. **25.** $x^{2n-2}x^{5n+3}$. **26.** $b^{m+1}b^{n-1}$. **27.** $a^{3n}a^{5m}$.

Degree. Homogeneous Expressions.

25. An integral term which is the product of n letters is said to be of the *nth degree.*

Thus, the **Degree of an Integral Term** is indicated by the sum of the exponents of its literal factors.

E.g., $3ab$ is of the *second* degree; $2x^2y, = 2xxy$, is of the third degree.

The **Degree of a Multinomial** is the degree of that term which is of highest degree.

E.g., the degree of $x^2y + xy^3 - x^2y^3z$ is the degree of x^2y^3z; *i.e.*, the *sixth.*

26. It is often desirable to speak of the degree of a term, or of an expression, in regard to one or more of its literal factors.

E.g., the term ax^2y^3 is of the *fifth* degree in x and y, of the *first* degree in a, of the *second* degree in x, of the *third* degree in y, etc.

The expression $ax^2 + 2\,bxy + cy^2$ is of the *second* degree in x, in y, and in x and y.

27. A **Homogeneous Expression** in one or more letters is an expression all of whose terms are of the same degree in these letters.

E.g., $a^2 + 2\,ab + b^2$ is homogeneous in a and b.

28. If the terms of a multinomial be arranged so that the exponents of some one letter increase, or decrease, from term to term, the multinomial is said to be arranged to *ascending*, or *descending*, powers of that letter. The letter is called the *letter of arrangement*.

E.g., $x^4 - 3\,x^3y^2 + 2\,x^2y + xy^3$ is arranged to *descending* powers of x, which is then the letter of arrangement; or, when written $x^4 + 2\,x^2y - 3\,x^3y^2 + xy^3$, to *ascending* powers of y, which is then the letter of arrangement.

EXERCISES X.

What is the degree of $2\,a^3b^2x^4y^5$

1. In a? **2.** In x? **3.** In b and y? **4.** In a, b, x, and y?

What is the degree of the expression $a^3x^4 - 6\,a^2b^2x^3y + 5\,abx^2y^3$

5. In x? **6.** In y? **7.** In a? **8.** In b?

9. Arrange $2\,x - 3\,x^5 + 7 - 2\,x^4 + 3\,x^2$ to ascending powers of x; to descending powers of x.

10. Arrange $3\,y - 7\,xy^3 + 5\,x^3y^2 + 4\,x^2y^4$ to ascending powers of x; to ascending powers of y.

Multiplication of Monomials by Monomials.

29. Ex. 1. $$3a \times 5b = 3 \times 5 \times a \times b,$$
$$= 15ab.$$

Ex. 2. $$2x \times (-4y^2) = 2(-4)xy^2 = -8xy^2.$$

Ex. 3. $$\tfrac{2}{3}a^3 \times 6ab^2 \times 11b^5 = \tfrac{2}{3} \times 6 \times 11 \times a^3ab^2b^5 = 44a^4b^7.$$

Ex. 4. $$-3x^m \times 4x^2 = -3 \times 4 \times x^mx^2 = -12x^{m+2}.$$

Ex. 5. $$5x^{n+1} \times 7x^{n-1} = 5 \times 7 \times x^{n+1}x^{n-1} = 35x^{n+1+n-1} = 35x^{2n}.$$

These examples illustrate the following method of multiplying two or more monomials.

Multiply the product of the numerical coefficients by the product of the literal factors.

EXERCISES XI.

Multiply:

1.	2.	3.	4.	5.	6.
$2a$	$3x$	$-5x^2$	$-6m$	$5a$	$-5m^3$
3	-2	6	-8	$-2b$	$7n$

7.	8.	9.	10.	11.	12.
$5x$	$-x^2y$	$7a^2$	$-5x^2y$	$-72a^2bc$	$5x^2yz^3$
$-6x^2$	$-xy^2$	$-3ab$	$-2xy^3$	$2abc^2$	$-3xy^2z^2$

13. $2(a+b)$ by $3(a+b)^2$. **14.** $-5(x-y)^2$ by $3a(x-y)^3$.

Simplify the following continued products

15. $3ab \times 5bc \times 6ac$. **16.** $-7x^2y \times (-2y^2z) \times 3xz^2$.

17. $-xy^2 \times 7bx^2z \times 2bx^2yz$. **18.** $x^2y^{n+1} \times 5x^my^{2n} \times (-x^{5n}y^{2n-1})$.

Multiply:

19. $2a^3b^2c,\ -3ab^2c^3,\ a^4b^4c,\ -5abc^4$.

20. $a^{m+2},\ a^{2m},\ a^{3-m},\ a^{m-n},\ a^{2n-3m}$.

21. $x^{n+1},\ -5x^{n-1},\ 2x^{2-m},\ x^{m+2}$.

Multiplication of a Multinomial by a Monomial.

30. If the indicated operation within the parentheses in the product, $4(2+3-1)$, be first performed, we have

$$4(2+3-1)=4\times 4=16.$$

But if each term within the parentheses be multiplied by 4 and the resulting products be then added, we have

$$4\times 2+4\times 3-4\times 1=8+12-4=16, \text{ as above.}$$

Therefore $4(2+3-1)=4\times 2+4\times 3-4\times 1.$

This example illustrates the following method of multiplying a multinomial by a monomial:

Multiply each term of the multinomial by the monomial, and add algebraically the resulting products. That is,

$$\boldsymbol{a(b+c-d)=ab+ac-ad.}$$

This principle is called the **Distributive Law** for multiplication.

31. Ex. **1.** Multiply $(x-y)$ by 3.

We have $3(x-y)=3x-3y.$

Ex. **2.** Multiply $3x-2y-7z$ by $-4x$.

We have

$$-4x(3x-2y-7z)=(-4x)(3x)-(-4x)(2y)-(-4x)(7z)$$
$$=-12x^2+8xy+28xz.$$

Such steps as changing $(-4x)(3x)$ into $-12x^2$, $-(-4x)(2y)$ into $+8xy$, and $-(-4x)(7z)$ into $+28xz$, should be performed mentally.

The work may be arranged as in arithmetic, by placing the multiplier under the multiplicand:

$$\begin{array}{r} 3x-2y-7z \\ -4x \\ \hline -12x^2+8xy+28xz \end{array}$$

It is customary to multiply from left to right, instead of *from right to* left as in arithmetic.

EXERCISES XII.

Multiply:

1. $x+1$ by 3. 2. $a-3$ by 5. 3. $2m+5$ by -3.

4. $3x-7$ by -8. 5. $2a+3b$ by $3a$. 6. $5x-3y$ by $2x$.

7. $6a^2-5b$ by $5b$. 8. $3x-5y^2$ by $-6xy$.

9. $8x^2+5y^2$ by $2xy$. 10. $a+b-c$ by 5.

11. $x-y-z$ by -3. 12. $3a+2b-5c$ by 4.

13. $5m-3n-4p$ by -3. 14. $2a-7b+3c$ by $-5a$.

15. $-5x^2+3y^2-2z^3$ by $-2xyz$.

Multiply a^2-3a+1 by

16. $2a$. 17. $-3b$. 18. $5ab$. 19. $-6a^2b^3$.

Multiply $x^2y+3xy-5y^2$ by

20. $-3x^2$. 21. $-5y^2$. 22. $-6x^2y$. 23. $5x^2y^2$.

Simplify the result of substituting $a+b-c$ for x, and $a-b+c$ for y, in the following expressions:

24. $5bx-7ay$. 25. $3a^2bx-14ab^2y$. 26. $7abx+2bcy$.

Find the values of the results of Exx. 24–26

27. When $a=-2,\ b=3,\ c=-4$.

28. When $a=5,\ b=-7,\ c=-5\frac{1}{2}$.

Multiply $5x^n-3x^{n-3}y^2+4x^{n-5}y^4+y^{n-4}$ by

29. x^3. 30. $-5x^2y$. 31. $3x^ny^4$. 32. $-6\frac{4}{5}x^ny^m$.

Simplify the following expressions:

33. $4x-2\{[x-3(2-x)]x-4\}$.

34. $13a-13\{10[7(4a-3)-6]-9a\}$.

35. $-206-2\{x-5[3-2x-6(4x-7)]-3(5-2x)\}$.

36. $\{[(x+y^2)x-(2y-1)]x-(x^2-2y)x-x^2y^2\}^2$.

Multiplication of Multinomials by Multinomials.

32. Ex. Multiply $7-5$ by $2+3$.

If we let a stand for $7-5$, we have

$$(2+3)a = 2a + 3a.$$

Now replacing a by $7-5$, we obtain

$$\begin{aligned}(2+3)(7-5) &= 2(7-5)+3(7-5)\\ &= 2\times 7 - 2\times 5 + 3\times 7 - 3\times 5.\end{aligned}$$

This example illustrates the following method of multiplying a multinomial by a multinomial:

Multiply each term of the multiplicand by each term of the multiplier, and add algebraically the resulting products.

In general,

$$\begin{aligned}(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{c}+\boldsymbol{d}-\boldsymbol{e}) &= \boldsymbol{a}(\boldsymbol{c}+\boldsymbol{d}-\boldsymbol{e})+\boldsymbol{b}(\boldsymbol{c}+\boldsymbol{d}-\boldsymbol{e})\\ &= \boldsymbol{ac}+\boldsymbol{ad}-\boldsymbol{ae}+\boldsymbol{bc}+\boldsymbol{bd}-\boldsymbol{be}.\end{aligned}$$

33. 1. Multiply $-3a+2b$ by $2a-3b$.

We have

$$\begin{array}{rrrr} & -3a & +\ 2b & \\ & 2a & -\ 3b & \\ \hline 2a(-3a+2b) = & -6a^2 & +\ 4ab & \\ -3b(-3a+2b) = & & +\ 9ab & -6b^2 \\ \hline & -6a^2 & +13ab & -6b^2 \end{array}$$

The work is arranged as follows: *Write the multiplier under the multiplicand; the first partial product,* i.e., *the product of the multiplicand by the first term of the multiplier, under the multiplier; the second partial product under the first; and so on, placing like terms of the partial products in the same column.*

Ex. 2. Multiply $x+a$ by $x+b$.

We have

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \qquad\ \ bx+ab \\ \hline x^2+(a+b)x+ab \end{array}$$

Ex. 3. **Multiply** $4a^2+1-2a-8a^3$ **by** $1+2a$.

Arranging to ascending powers of a, we have

$$\begin{array}{l}
1-2a+4a^2-8a^3 \\
1+2a \\
\hline
1-2a+4a^2-8a^3 \\
\phantom{1-{}}2a-4a^2+8a^3-16a^4 \\
\hline
1 \qquad\qquad\qquad\qquad -16a^4
\end{array}$$

Ex. 4. **Multiply** $x^2+y^2+1-xy-x-y$ **by** $x+y+1$.

Arranging to descending powers of x, we have

$$\begin{array}{llllllll}
x^2 & -xy & -x & +y^2 & -y & +1 & & \\
x & +y & +1 & & & & & \\
\hline
x^3 & -x^2y & -x^2 & +xy^2 & -xy & +x & & \\
 & x^2y & & -xy^2 & -xy & & +y^3-y^2 & +y \\
 & & & - & xy & -x & +y^2 & -y+1 \\
\hline
x^3 & & & & -3xy & & +y^3 & +1
\end{array}$$

Ex. 5. Multiply $2x^{m+1}-5x^m+7x^{m-1}$ by $x^{2m}-x^{2m-1}$.

We have

$$\begin{array}{lllll}
2x^{m+1} & -5x^m & +7x^{m-1} & & \\
x^{2m} & -x^{2m-1} & & & \\
\hline
2x^{3m+1} & -5x^{3m} & +7x^{3m-1} & & \\
 & -2x^{3m} & +5x^{3m-1} & -7x^{3m-2} & \\
\hline
2x^{3m+1} & -7x^{3m} & +12x^{3m-1} & -7x^{3m-2} &
\end{array}$$

EXERCISES XIII.

Multiply:

1. $a+1$ by $a+2$.
2. $x+1$ by $x-2$.
3. $m-5$ by $m+3$.
4. $y-6$ by $y-5$.
5. $m-12$ by $m-3$.
6. $a-12$ by $a-15$.
7. $2x+1$ by $x+3$.
8. $3a+5$ by $2a-3$.
9. $11m-6$ by $2m-5$.
10. $15x-8$ by $10x-3$.
11. $x+y$ by $x-y$.
12. $2a+b$ by $3a-b$.

13. $3m-2n$ by $5m+3n$.

14. $5x-6y$ by $3x-2y$.

15. $2x^2+7y$ by $5x^2-3y$.

16. $11m^2+6n$ by $5m^2-7n$.

17. $2a^2+3b^2$ by $4a^2-5b^2$.

18. $3x^2+4xy$ by $2x^2+3xy$.

19. $7a^2+2ab$ by $3a^2-5ab$.

20. $6x^2-5xy$ by $3x^2-2xy$.

21. x^2+x+1 by $x-1$.

22. x^2-x+1 by $x+1$.

23. a^2+5a-6 by $a-3$.

24. $x^2-11x+12$ by $x-8$.

25. $2a^2+3a-5$ by $3a-2$.

26. $5x^2-7x+2$ by $6x-7$.

27. $5x^2-2x+1$ by $5x+2$.

28. $3x^2+4x-5$ by $3x-4$.

29. $x^2+2xy+y^2$ by $x+y$.

30. $a^2-2ab+b^2$ by $a-b$.

31. x^3-x^2-x+1 by $x+1$.

32. x^3+x^2+x+1 by $x-1$.

33. $8x^3-4x^2+2x-1$ by $2x+1$.

34. $x^3+x^2y+xy^2+y^3$ by $x-y$.

35. $27a^3+18a^2b+12ab^2+8b^3$ by $3a-2b$.

36. $2a+3b+5c$ by $2a+3b-5c$.

37. $6x^2+3x+1$ by $6x^2-3x+1$.

38. $1+xy+x^2y^2$ by $1-xy-x^2y^2$.

39. $2a^2-3ab+5b^2$ by $2a^2+3ab-5b^2$.

40. $2x^2+3xy+4y^2$ by $3x^2-4xy+y^2$.

41. x^3-2x^2+3x-1 by x^2-3x+2.

42. x^4-5x^2+6x-3 by x^2+5x-4.

43. x^4-6x^3+2x+5 by $3x^2-2x+5$.

44. $x^3-4x^2y+2xy^2-y^3$ by $x^2-3xy+y^2$.

45. $x^4+2x^3+x^2-4x-11$ by x^2-2x+3.

46. $x^2-xy+y^2+x+y+1$ by $x+y-1$.

47. $x^n-2x^{n-1}-3x^{n-2}-5x^{n-3}$ by $x+1$.

48. $5x^n+3x^{n-1}-8x^{n-2}-3$ by $x-2$.

49. $a^{n+1}-5a^n+7a^{n-1}-3$ by a^2+a+1.

50. $x^{3n}-x^{2n}+x^n-1$ by x^n+1.

51. $a^{2n}-2a^nb^n+b^{2n}$ by $a^{2n}+2a^nb^n+b^{2n}$.

52. $(x+m)(x+n)$.
53. $(x-m)(x-n)$.
54. $(x+m)(x-n)$.
55. $(x-m)(x+n)$.
56. $[x^2-(a+b)x+ab](x-c)$.
57. $[x^2+(a-b)x-ab](x+c)$.

Simplify each of the following expressions:

58. $(x+4)(x-3)-(x+2)(x+6)$.
59. $(x+8)(x-4)-(x+16)(x+2)$.
60. $(x-2)(x+3)(x-4)-(x-3)(x-5)(x-7)$.
61. $(a+1)(a+2)(a+3)-(a-1)(a-2)(a-3)$.
62. $(a+b)^2-(a+c)^2-(b+c)^2$.
63. $(a+b+c)^3-3(a+b+c)(a^2+b^2+c^2)$.
64. $x^2(y-z)+y^2(z-x)+z^2(x-y)+(x-y)(y-z)(z-x)$
65. $(x-y)^3+(y-z)^3+(z-x)^3-3(x-y)(y-z)(z-x)$.
66. $(2m^2+3m-2)(m-1)(2m+3)$.
67. $(x^2+4x-1)(x^2-2x+1)(x+2)$.
68. $(x+y+z)(x+y-z)(z+x-y)$.
69. $(a^2-a+1)(a^2+a+1)(a^4-a^2+1)$.

Simplify each of the following expressions:

70. $(\frac{1}{2}a^2+\frac{1}{4}b^2)(\frac{1}{2}a^2+\frac{1}{4}b^2)$.
71. $(\frac{1}{2}a^2+\frac{1}{2}b^2)(\frac{1}{2}a^2-\frac{1}{2}b^2)$.
72. $(\frac{2}{3}a-\frac{2}{3}b+\frac{3}{4}c)(\frac{2}{3}a-\frac{3}{2}b+\frac{3}{4}c)$.
73. $(\frac{1}{2}x-\frac{2}{3}y+5z)(\frac{2}{3}x-3y-\frac{1}{2}z)$.
74. $(2\frac{1}{2}-3\frac{1}{4}x+3\frac{1}{2}x^2)(\frac{1}{4}x^2+2\frac{1}{2}x+1\frac{1}{3})$.
75. $(\frac{3}{4}a^2+\frac{1}{2}b^2-\frac{1}{9}c^2)(\frac{3}{4}a^2-\frac{1}{2}b^2-\frac{1}{9}c^2)$.
76. $(\frac{1}{2}ax+\frac{1}{3}bx^2+\frac{1}{4}cx^3)(\frac{1}{2}ax+\frac{1}{3}bx^2-\frac{1}{4}cx^3)$.

Zero in Multiplication.

34. Since $N\cdot 0=N(b-b)$, by definition of 0,

$$=Nb-Nb=0,$$

we have $\boldsymbol{N\cdot 0=0}$ and $\boldsymbol{0\cdot N=0}$.

That is, *a product is* 0 *if one of its factors be zero.*

EXERCISES XIV.

1. What is the value of $2(a-b)$, when $b=a$?

2. What is the value of $(a+b)(c-d)$, when $c=d$?

3. What is the value of $(b+c)(a+b-c)$, when $c=a+b$?

4. What is the value of $(x^2-9)(x^4-7x^3+2x-9)$, when $x=3$?

For what values of x does each of the following expressions reduce to 0:

5. $x(x-2)$? 6. $(x-4)(x+7)$? 7. $(x-1)(x-a)$?

8. $(x-6)(x+8)(x^2-25)$? 9. $x(x-a)(x-b)(x-c)$?

Equations and Problems.

35. Ex. Find the value of x from the equation

$$3(x-4)+5=4(x-3).$$

Removing parentheses, $3x-12+5=4x-12.$

Cancelling -12, $3x+5=4x.$

Transferring terms, $3x-4x=-5,$

or $-x=-5.$

Dividing by -1, $x=5.$

Check: $3(5-4)+5=4(5-3)$, or $3+5=4\times 2$, or $8=8$.

To solve such equations: *Remove parentheses, and proceed as in Art.* 22.

Pr. A number of persons were to raise a fund by paying \$5 each. Had there been 4 persons more, each would have had to contribute only \$3. How many persons were there?

Let x stand for the number of persons.

Then the number of dollars contributed was $5x$.

Had there been 4 persons more, there would have been $x+4$ persons.

Then the number of dollars contributed would have been $3(x+4)$.

The problem implies,

in *verbal* language: *the number of dollars contributed in the one case is equal to the number of dollars contributed in the other;*

in *algebraic* language: $5x = 3(x + 4)$.

Removing parentheses, $5x = 3x + 12$.

Transferring $3x$, $2x = 12$.

Dividing by 2, $x = 6$.

Check: 6 persons contributed $6 \times 5, = 30$ dollars; $6 + 4$, or 10, persons would have contributed $10 \times 3, = 30$ dollars.

EXERCISES XV.

Solve the following equations:

1. $5(x + 1) = 6$. **2.** $4(2x - 1) = 5$.

3. $3(x + 5) + 17 = 26$. **4.** $14 + 3(7 - 2x) = 29$.

5. $15 + 4(8 - 2x) = 7$. **6.** $25 - 3(5 - 4x) = 22$.

7. $27 + 4(2x - 8) = 12$. **8.** $11(2 - 5x) = 47 - 30x$.

9. $12(4x - 5) = 13 - 98x$. **10.** $7x - 6(10 - x) = 33x$.

11. $4(2x + 3) - 3(2x + 4) = 10$.

12. $5(3x + 4) - 2(4x - 3) = 54$.

13. $7(2x - 3) - 11(5x - 4) = 64$.

14. $(x - 3)(x - 4) = x^2 + 5$.

15. $(x - 4)(x - 6) = x(x - 9)$.

16. $(x + 1)(x + 2) = (x - 3)(x - 4)$.

17. $6x(2x + 3) = (3x + 2)(4x + 3)$.

18. The sum of two numbers is 50. If five times the less exceeds three times the greater by 10, what are the numbers?

19. Two boys, A and B, had the same number of apples. A said to B: "Give me 5 apples and I shall have twice as many as you will have left." How many apples had each?

20. Add 10 to a certain number, and multiply the sum by 2, or subtract 8 from the same number, and multiply the difference by 5. The results will be equal. What is the number?

21. A is 30 years old, and B is 12 years old. After how many years will A be twice as old as B?

22. A father is 30 years older than his son; 5 years ago he was four times as old. What are the ages of father and son?

23. A and B invested equal amounts. A gained $ 200, and B gained $ 2600. If B then had three times as much as A, how much did each invest?

24. Three boys, A, B, and C, catch 128 fish. If B catches 10 more fish than A, and C catches three times as many as A and B together, how many fish does each boy catch.

25. In one room there are twice as many persons as in a second room. If 10 persons pass from the first room into the second, there will be three times as many persons in the second as in the first. How many persons are there in each room?

26. A woman has enough money to buy 11 yards of cloth of one kind, or 8 yards of another kind. If the latter costs 30 cents more a yard than the former, how much does a yard of each kind cost?

27. In a stairway there are 45 steps of a certain height. If the steps had been made 1 inch higher, there would have been only 40. How high are the steps?

28. The capacity of a certain vessel is 90 gallons. One pipe lets in 2 gallons a minute and a second pipe 1 gallon. If the first pipe is opened 15 minutes before the second, how long after the first pipe is opened will the vessel be filled?

29. A farmer has two fields containing together 5 acres. A offers to pay $ 62 an acre for the first field and $ 72 an acre for the second. B offers to pay $ 60 an acre for the first field and $ 75 an acre for the second. If both offers amount to the same, how many acres are there in each field?

30. The capacity of a certain cistern is 2200 gallons. One pipe lets in 80 gallons in a minute, and a second pipe 50 gallons. How many minutes must the first pipe be opened before the second in order that the cistern may be filled 4 minutes after the second pipe is opened?

31. One cask contains 70 gallons, and another 50 gallons. If three times as many gallons are drawn from the larger as from the smaller, the contents of the smaller will be equal to three times the contents of the larger. How many gallons are drawn from each cask?

32. A man has $115 in two-dollar bills and five-dollar bills. If he has 35 bills altogether, how many of each kind has he?

33. A rides his bicycle 12 miles an hour, and B his 10 miles an hour. A rides a certain number of hours, and B rides 2 hours longer. If they ride the same distance, how many hours does each ride?

34. Twenty-five men were to raise a certain fund by contributing equal amounts. But 5 men failed to contribute, and in consequence each of the remaining men had to contribute $2 more. What was to be the original contribution of each? What was the amount of the fund?

DIVISION.

36. One power is said to be *higher* or *lower* than another according as its exponent is *greater* or *less* than the exponent of the other.

E.g., a^4 is a higher power than a^3 or b^2, but is a lower power than a^6 or b^7.

Quotient of Powers of One and the Same Base.

37. Ex. $a^7 \div a^3 = (aaaaaaa) \div (aaa).$

$= (aaaa) \times (aaa) \div (aaa)$, by Ch. II, Art. 16.

$= aaaa = a^4 = a^{7-3}.$

This example illustrates the following method of dividing a higher power by a lower power of the same base:

The exponent of the quotient is the exponent of the dividend minus the exponent of the divisor; or, stated symbolically,

$$a^m \div a^n = a^{m-n}.$$

We also have

$$a^m \div a^n = 1, \text{ when } m = n.$$

E.g., $a^2 \div a^2 = 1.$

EXERCISES XVI.

Express each of the following quotients as a single power:

1. $2^3 \div 2.$ **2.** $3^5 \div 3^2.$ **3.** $x^3 \div x^2.$ **4.** $a^6 \div a^4.$

5. $x^7 \div x^3.$ **6.** $a^6 \div a^5.$ **7.** $(-a)^6 \div a^5.$ **8.** $(3x)^5 \div (3x).$

9. $(ab)^7 \div (-ab)^4.$ **10.** $5^n \div 5^3.$ **11.** $a^{n+1} \div a.$

12. $x^{n+7} \div x^n.$ **13.** $x^{a+3} \div x^{a+1}.$ **14.** $a^{2n} \div a^{n-1}.$

Division of Monomials by Monomials.

38. Ex. **1.** $12a \div 4 = (12 \div 4) \times a = 3 \times a = 3a.$

Ex. **2.** $-27x^7 \div 3x^2 = (-27 \div 3) \times (x^7 \div x^2) = -9 \times x^5 = -9x^5.$

Ex. **3.** $15a^3b^2 \div (-5ab^2) = [15 \div (-5)] \times (a^3 \div a) \times (b^2 \div b^2)$
$= -3a^2.$

Ex. **4.** $-16x^{2m}y^{n+1} \div (-8x^my^{n-1})$
$= [-16 \div (-8)] \times (x^{2m} \div x^m) \times (y^{n+1} \div y^{n-1})$
$= 2x^{2m-m}y^{n+1-(n-1)} = 2x^my^2.$

These examples illustrate the following method of dividing a monomial by a monomial:

Multiply the quotient of the numerical coefficients by the quotient of the literal factors.

EXERCISES XVII.

Divide

1. $2\underline{)6a}.$ **2.** $5\underline{)-10x}.$ **3.** $4\underline{)-16m}.$ **4.** $-5y\underline{)-25y}.$ **5.** $-7m\underline{)-49my}.$

6. $5x^2$ by x.

7. $-6a^3$ by $2a$.

8. $25m^5$ by $-5m^2$.

9. $-4a^2b$ by $-2a$.

10. $6abc^3$ by $-3ac$.

11. $-9a^3b$ by $3a^2b$.

12. $30x^8y^4$ by $5x^4y^2$.

13. $-15a^5b^7$ by $-3a^3b^5$.

14. $35a^7b^{10}c^{13}$ by $-5a^4b^5c^6$.

15. $12m^6n^7p^8$ by $-2m^2n^4p^6$.

16. $15(a+b)$ by $3(a+b)$.

17. $25x^2(x+1)^3$ by $-5x(x+1)^2$.

18. $10a^{2n}b^5$ by $-5a^nb^3$.

19. $-27x^{n+1}y^{3m}$ by $-9xy^{2m}$.

20. $x^{2n-1}y^{3m+2}$ by $x^{n+1}y^{2m-3}$.

21. $a^{n-1}b^{n-2}$ by $a^{n-3}b^{n-4}$.

Simplify

22. $a^3x^5 \div (-ax^3) \times 2axy$.

23. $35x^2y^3z \times 2xy^3 \div (7x^2y^2z)$.

24. $6x^{m+1}y^{n-1} \div (-x^{m-1}y^{m-n}) \times (3x^2y)$.

Division of a Multinomial by a Monomial.

39. If the indicated operation within the parentheses in the quotient $(8+6-4) \div 2$ be first performed, we have

$$(8+6-4) \div 2 = 10 \div 2 = 5.$$

But if each term within the parentheses be first divided by 2, and the resulting quotients be then added, we have

$$8 \div 2 + 6 \div 2 - 4 \div 2 = 4 + 3 - 2 = 5, \text{ as above.}$$

Therefore $(8+6-4) \div 2 = 8 \div 2 + 6 \div 2 - 4 \div 2$.

This example illustrates the following method of dividing a multinomial by a monomial:

Divide each term of the multinomial by the monomial, and add algebraically the resulting quotients.

That is,

$$(b+c-d) \div a = b \div a + c \div a - d \div a.$$

This principle is called the **Distributive Law** for division.

40. Ex. 1. Divide $6x^2 - 12x$ by $3x$.

We have $(6x^2 - 12x) \div 3x = 6x^2 \div 3x - 12x \div 3x = 2x - 4$.

Ex. 2. $(4a^{2m-1} - 8a^{3m+1}) \div 4a^{m-1}$

$$= 4a^{2m-1} \div 4a^{m-1} - 8a^{3m+1} \div 4a^{m-1} = a^m - 2a^{2m+2}.$$

EXERCISES XVIII.

Divide

1. $5 + 10\,a$ by 5.
2. $4\,a + 8\,b$ by -4.
3. $ax + bx$ by x.
4. $3\,a^2 - 6\,ab$ by $-3\,a$.
5. $21\,a^2b - 14\,ab^2$ by $-7\,ab$.
6. $8\,am^2 - 2\,a^2m + 4\,a^3m^2$ by $2\,am$.
7. $25\,(a+b)^3 - 20\,(a+b)$ by $5\,(a+b)$.
8. $2\,(x-y)^3 - 12\,a\,(x-y)^4 - 6\,(x-y)^6$ by $2\,(x-y)^2$.

Simplify

9. $2\,a^3 - (a^3 - 3\,a) \div a$.
10. $(6\,x - 4\,x^2) \div 2\,x - (-2\,x^2y + 3\,xy) \div xy$.
11. $(ab - a^2b + 3\,a^3b) \div ab - (4\,a^3 - 4\,a^2) \div 2\,a$.

Divide $9\,a^2x^6 - 6\,a^3x^4 + 12\,a^5x^3$ by

12. $3\,a^2$.
13. $-3\,x^3$.
14. ax^2.
15. $-\frac{3}{2}\,a^2x^3$.

Divide $105\,a^3b^2c^4 - 21\,a^4b^3c^3 + 42\,a^5b^4c^3$ by

16. $7\,a^3$.
17. $-3\,a^3b^2$.
18. $-a^2bc^3$.
19. $\frac{3}{4}\,a^2b^2c^2$.

Divide $15\,x^{2n+1}y^5 - 12\,x^{2n+3}y^3 - 18\,x^{2n+5}y^4$ by

20. $3\,x^n$.
21. $-5\,x^{n+1}y^2$.
22. $-3\,x^{2n+1}y$.
23. $\frac{1}{2}\,x^{2n-5}y^3$.

Zero in Division.

41. Since $0 \div N = (a - a) \div N$, by definition of 0,

$$= a \div N - a \div N = 0.$$

We have $\mathbf{0 \div \mathit{N} = 0}$, when $\boldsymbol{N}$ is not equal to **0**.

Observe that this relation is proved only when N is not equal to 0.

Division of a Multinomial by a Multinomial.

42. The division of one multinomial by another is performed in a way similar to that of dividing one number by another in Arithmetic.

Ex. Divide 125 by 5.

We have

$$\begin{array}{rr|l}
 & 125 & 5 \\
20 \times 5 = & \underline{100} & 20+5 \;\; = 25 \\
125 - 20 \times 5 = & 25 & \\
 & \underline{25} &
\end{array}$$

The work is equivalent to the following:

$$125 \div 5 = 20 + (125 - 20 \times 5) \div 5 = 20 + 25 \div 5 = 25.$$

43. The number 20, obtained by the first step of the division, is called the **Partial Quotient** at that stage. It is the greatest number whose product by the divisor is equal to or less than the dividend.

In general, if D be the given *dividend*, d the given divisor, and q the *partial quotient*, the principle used above, stated symbolically, is:

$$\boldsymbol{D \div d = q + (D - qd) \div d.}$$

44. The following example illustrates the application of this principle in dividing one multinomial by another.

Ex. Divide $x^2 + 3x + 2$ by $x + 1$.

We have

$$\begin{aligned}
(x^2+3x+2) \div (x+1) &= x + [(x^2+3x+2) - x(x+1)] \div (x+1) && (1)\\
&= x + (x^2+3x+2-x^2-x) \div (x+1) && (2)\\
&= x + (2x+2) \div (x+1) && (3)\\
&= x+2+[(2x+2)-2(x+1)] \div (x+1) && (4)\\
&= x+2+0 \div (x+1) \\
&= x+2, \text{ since } 0 \div (x+1) = 0.
\end{aligned}$$

We take the quotient of the term containing the highest power of x in the dividend by the term containing the highest power of x in the divisor as the partial quotient at each step.

The work may be arranged more conveniently thus:

$$\begin{array}{l|l}
x^2 + 3x + 2 & \underline{x + 1} \\
& x + 2, \text{ quotient.} \\
\underline{x^2 + \quad x} & \cdots x(x+1) \text{ to be subtracted from } x^2 + 3x + 2;\text{ see (1) and (2) above.} \\
\quad 2x + 2 & \cdots \text{Remainder to be divided by } x + 1;\text{ see (3) above.} \\
\quad \underline{2x + 2} & \cdots 2(x+1) \text{ to be subtracted from } 2x + 2;\text{ see (4).}
\end{array}$$

45. The method of applying the principle of Art. 43 to the division of multinomials, as illustrated by this example, may be stated as follows:

Arrange the dividend and divisor to ascending or descending powers of some common letter, the letter of arrangement.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply the divisor by this first term of the quotient, and subtract the resulting product from the dividend.

Divide the first term of the remainder by the first term of the divisor, and write the result as the second term of the quotient.

Multiply the divisor by this second term of the quotient, and subtract the product from the remainder previously obtained. Proceed with the second remainder and all subsequent remainders, in like manner, until a remainder zero is obtained, or until the highest power of the letter of arrangement in the remainder is less than the highest power of that letter in the divisor.

In the first case the division is exact; in the second case the quotient at this stage of the work is called the quotient of the division, and the remainder the remainder of the division.

46. Ex. 1. Divide $x^2 - 4x - 5$ by $x - 5$. We have

$$\begin{array}{r|l}
x^2 - 4x - 5 & \underline{x - 5} \\
\underline{x^2 - 5x \quad\;\;} & x + 1 \\
x - 5 & \\
\underline{x - 5} &
\end{array}$$

Ex. 2. Divide

$$a^3b - 15\,b^4 + 19\,ab^3 + a^4 - 8\,a^2b^2 \text{ by } a^2 - 5\,b^2 + 3\,ab.$$

Arranging to descending powers of a, we have

$$\begin{array}{l|l}
a^4 + a^3b - 8\,a^2b^2 + 19\,ab^3 - 15\,b^4 & a^2 + 3\,ab - 5\,b^2 \\ \cline{2-2}
\underline{a^4 + 3\,a^3b - 5\,a^2b^2} & a^2 - 2\,ab + 3\,b^2 \\
\quad - 2\,a^3b - 3\,a^2b^2 + 19\,ab^3 & \\
\quad \underline{- 2\,a^3b - 6\,a^2b^2 + 10\,ab^3} & \\
\qquad\qquad 3\,a^2b^2 + 9\,ab^3 - 15\,b^4 & \\
\qquad\qquad \underline{3\,a^2b^2 + 9\,ab^3 - 15\,b^4} &
\end{array}$$

Ex. 3. Divide $8\,x^3 - y^3$ by $2\,xy + 4\,x^2 + y^2$.

Arranging the divisor to descending powers of x, we have:

$$\begin{array}{l|l}
8\,x^3 - y^3 & 4\,x^2 + 2\,xy + y^2 \\ \cline{2-2}
\underline{8\,x^3 + 4\,x^2y + 2\,xy^2} & 2\,x - \quad y \\
\quad - 4\,x^2y - 2\,xy^2 - y^3 & \\
\quad \underline{- 4\,x^2y - 2\,xy^2 - y^3} &
\end{array}$$

Observe that the remainder after the first partial division is arranged to descending powers of x.

Ex. 4. Divide $12\,a^{n+1} + 8\,a^n - 45\,a^{n-1} + 25\,a^{n-2}$ by $6\,a - 5$.

We have

$$\begin{array}{l|l}
12a^{n+1} + 8\,a^n - 45\,a^{n-1} + 25\,a^{n-2} & 6\,a - 5 \\ \cline{2-2}
\underline{12\,a^{n+1} - 10\,a^n} & 2\,a^n + 3\,a^{n-1} - 5\,a^{n-2} \\
\qquad 18\,a^n - 45\,a^{n-1} & \\
\qquad \underline{18\,a^n - 15\,a^{n-1}} & \\
\qquad\qquad - 30\,a^{n-1} + 25\,a^{n-2} & \\
\qquad\qquad \underline{- 30\,a^{n-1} + 25\,a^{n-2}} &
\end{array}$$

Ex. 5. Divide $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$ by $x^2 + (a + b)x + ab$.

We have

$$\begin{array}{l|l}
x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc & x^2 + (a + b)x + ab \\ \cline{2-2}
\underline{x^3 + (a + b \quad\;\;)x^2 + \qquad\qquad\quad abx} & x + c \\
\qquad\qquad cx^2 + (ac + bc)\,x + abc & \\
\qquad\qquad \underline{cx^2 + (ac + bc)\,x + abc} &
\end{array}$$

EXERCISES XIX.

Find the values of the following indicated division:

1. $(x^2+2x+1)\div(x+1)$.
2. $(x^2+11x+30)\div$
3. $(x^2+x-90)\div(x-9)$.
4. $(x^2-5x+6)\div($
5. $(x^2+7x-44)\div(x+11)$.
6. $(x^2-3x-40)\div($
7. $(3x^2-13x-10)\div(3x+2)$.
8. $(2a^2+a-6)\div(2$
9. $(15x^2-7x-2)\div(5x+1)$.
10. $(6x^2-23x+20)\div$
11. $(x^3-4x^2-20x+3)\div(x+3)$.
12. $(x^3-7x^2+13x-15)\div(x-5)$.
13. $(4x^3-3x^2-24x-9)\div(x-3)$.
14. $(3x^3-13x^2+23x-21)\div(3x-7)$.
15. $(18x^3+7x+10)\div(3x+2)$.
16. $(50x^3-23x+6)\div(5x-2)$.
17. $(a^2+2ab+b^2)\div(a+b)$.
18. $(2x^2+6a^2+7ax)\div(2x+3a)$.
19. $(35x^2-88y^2+xy)\div(7x-11y)$.
20. $(a^2-18axy-243x^2y^2)\div(a+9xy)$.
21. $(8x^2y^2-65xyz^2-63z^4)\div(xy-9z^2)$.
22. $(6n^3-7n^2x+2nx^2)\div(-x+2n)$.
23. $(x^4y+6x^5-2x^3y^2)\div(3x^2+2xy)$.
24. $(3x^4-3x^3-2x^2-x-1)\div(3x^2+1)$.
25. $(a^6-6a^4+9a^2-4)\div(a^2-1)$.
26. $(21a^6b+20b^4-22a^2b^3-29a^4b^2)\div(3a^2b-$
27. $(x^3+8x^2+9x-18)\div(x^2+5x-6)$.
28. $(x^4+x^3-4x^2+5x-3)\div(x^2+2x-3)$.
29. $(6x^4-x^3-11x^2-10x-2)\div(2x^2-3x-$
30. $(x^3-1)\div(x^2+x+1)$.
31. $(a^3+8)\div(a^2-$
32. $(x^6-64y^3)\div(x^2-4y)$.
33. $(a^5x^5+y^5)\div(ax$

34. $(x^4 + x^2 + 1) \div (x^2 - x + 1)$.

35. $(a^4x^5 + 64\,x) \div (4\,ax + a^2x^2 + 8)$.

36. $(4\,a^4 - 25\,c^4 - 30\,b^2c^2 - 9\,b^4) \div (2\,a^2 + 5\,c^2 + 3\,b^2)$.

37. $(27\,x^4 - 6\,c^2x^2 + \frac{1}{3}\,c^4) \div (c^2 - 6\,cx + 9\,x^2)$.

38. $(8\,a^3n^3 + 32\,a^6 + \frac{1}{2}\,n^6) \div (4\,an + n^2 + 4\,a^2)$.

39. $(16\,a^4b^2 + 9\,a^2b^4 - 12\,a^3b^3 - 8\,a^5b + 3\,a^6) \div (a^4 + 3\,a^2b^2 - 2\,a^3b)$.

40. $(28\,a^5c - 26\,a^3c^3 - 13\,a^4c^2 + 15\,a^2c^4) \div (2\,a^2c^2 + 7\,a^3c - 5\,ac^3)$.

41. $(81\,z^8 - 90\,b^4z^4 + 81\,b^6z^2 - 20\,b^8) \div (9\,z^4 + 9\,b^2z^2 - 5\,b^4)$.

42. $(x^3 + y^3 + 3\,xy - 1) \div (x + y - 1)$.

43. $(a^3 + b^3 + c^3 - 3\,abc) \div (a + b + c)$.

44. $(a^2 + 2\,ab + b^2 - x^2 + 4\,xy - 4\,y^2) \div (a + b - x + 2\,y)$.

45. $(a^2 + 2\,ac - b^2 - 2\,bd + c^2 - d^2) \div (a + c - b - d)$.

Find the values of the following indicated divisions:

46. $[x^2 + (a + 1)\,x + a] \div (x + a)$.

47. $[x^2 - (a + b)\,x + ab] \div (x - b)$.

48. $[cx^2 - (abc + 1)\,x + ab] \div (x - ab)$.

49. $[(b + c)\,x^2 - bcx + x^3 - bc\,(b + c)] \div (x^2 - bc)$.

50. $[x^3 + (a + b + c)\,x^2 + (ab + ac + bc)\,x + abc] \div (x + b)$.

51. $[x^3 - (a + b + c)\,x^2 + (ab + ac + bc)\,x - abc] \div (x - c)$.

52. $(6\,x^{3n} - 25\,x^{2n} + 27\,x^n - 5) \div (2\,x^n - 5)$.

53. $(6\,x^{5n} - 11\,x^{4n} + 23\,x^{3n} + 13\,x^{2n} - 3\,x^n + 2) \div (3\,x^n + 2)$.

54. $(6\,x^{2n+1} - 29\,x^{2n} + 43\,x^{2n-1} - 20\,x^{2n-2}) \div (2\,x^n - 5\,x^{n-1})$.

55. $(1 + a^{6x} - 2\,a^{3x}) \div (3\,a^{2x} + 2\,a^{3x} + 2\,a^x + a^{4x} + 1)$.

56. $(\frac{1}{2}\,x^2 + \frac{7}{6}\,xy - y^2) \div (\frac{2}{3}\,x + 2\,y)$.

57. $(x^2 + \frac{91}{80}\,xy - 7\,x - y^2 + 13\,y - 30) \div (\frac{2}{8}\,x - \frac{1}{5}\,y + 2)$.

58. $(-\frac{9}{16}\,x^6 + a^2x^4 - \frac{4}{9}\,a^4x^2 + \frac{1}{4}\,a^6) \div (\frac{3}{4}\,x^3 - \frac{2}{3}\,a^2x + \frac{1}{2}\,a^3)$.

59. $(\frac{4}{9}\,a^4 + \frac{9}{16}\,b^4 + a^2b^2 - \frac{16}{25}\,c^4) \div (\frac{2}{3}\,a^2 + \frac{3}{4}\,b^2 - \frac{4}{5}\,c^2)$.

47. In the equation $D \div d = q + (D - qd) \div d,$

$D - qd$ is the remainder at any stage of the work, and q is the corresponding partial quotient. If, for brevity, we let R stand for the remainder at any stage, we have

$$\boldsymbol{D \div d = q + R \div d.} \tag{1}$$

That is, *the result of dividing one number by another is equal to the partial quotient at any stage, plus the remainder at this stage divided by the given divisor.*

E.g., $$29 \div 6 = 4 + 5 \div 6 = 4 + \tfrac{5}{6};$$

$$(x^2 - x + 2) \div (x + 1) = (x - 2) + 4 \div (x + 1).$$

48. If both members of the equation

$$D \div d = q + R \div d$$

be multiplied by d, we have

$$\begin{aligned} D \div d \times d &= (q + R \div d)d \\ &= qd + R \div d \times d \\ &= qd + R, \text{ since } \div d \times d = \div 1. \end{aligned}$$

Therefore, $$\boldsymbol{D = qd + R.}$$

That is, *the dividend is equal to the product of the quotient at any stage and the divisor, plus the remainder at this stage.*

E.g., $29 = 4 \times 6 + 5$, and $x^2 - x + 2 = (x - 2)(x + 1) + 4.$

EXERCISES XX.

Find the remainder of each of the following indicated divisions, and verify the work by applying the principle of Art. 48:

1. $(x^2 - 7x + 11) \div (x - 2)$. **2.** $(3x^2 + 5x - 9) \div (x - 4)$.

3. $(x^3 - 17x^2 + 15x - 13) \div (2x - 5)$.

4. $(5x^5 - 7x^2 + 2x - 1) \div (x^2 - 7x + 3)$.

CHAPTER IV.

INTEGRAL ALGEBRAIC EQUATIONS.

We will now distinguish between two kinds of equations.

Identical Equations.

1. An example of the one kind is:

$$(a+b)(a-b)=a^2-b^2.$$

The first member is reduced to the second member by performing the indicated multiplication.

2. Such an equation is called an **Identical Equation,** or more simply, an **Identity.**

3. Notice that identical equations are true for all values that may be substituted for the literal numbers involved.

E.g., if $a=5$ and $b=3$, the above equation becomes

$$8\times 2=25-9, \text{ or } 16=16.$$

Conditional Equations.

4. An example of the second kind is:

$$x+1=3.$$

The first member of this equation reduces to the second member, when $x=2$. It seems evident that $x+1$ reduces to 3 *only* when $x=2$.

5. Such equations *impose conditions* upon the values of the literal numbers involved. Thus, the equation in Art. 4 imposes the condition that if 1 be added to the value of x, the sum will be 3.

A **Conditional Equation** is an equation one of whose members can be reduced to the other only for certain definite values of one or more letters contained in it.

Whenever the word *equation* is used in subsequent work we shall understand by it a *conditional equation,* unless the contrary is expressly stated.

6. An **Integral Algebraic Equation** is an equation whose members are integral algebraic expressions in an unknown number or unknown numbers.

E.g., $3x^2 - 4 = 2x$, and $\frac{2}{3}x + 5y = \frac{4}{5}$ are integral equations.

7. The **Degree** of an integral equation is the degree of its term of highest degree in the unknown number or numbers.

8. A **Linear** or **Simple Equation** is an equation of the *first* degree.

E.g., $x + 1 = 6$ is a linear equation in one unknown number.

9. A **Solution** of an equation is a value of the unknown number, or a set of values of the unknown numbers, which, if substituted in the equation, converts it into an identity.

E.g., 2 is a solution of the equation $x + 1 = 3$,
since, when substituted for x in the equation, it converts the equation into the identity $2 + 1 = 3$.

The set of values 1 and 2, of x and y, respectively, is a solution of the equation $x + y = 3$, since $1 + 2 = 3$ is an identity.

10. To **Solve** an equation is to find its solution.

An equation is said *to be satisfied by its solution,* or *the solution is said to satisfy the equation,* since it converts the equation into an identity.

11. When the equation contains only one unknown number, a solution is frequently called a **Root** of the equation.

E.g., 2 is a root of the equation $x + 1 = 3$.

Equivalent Equations.

12. Consider the solution of the equation

$$\tfrac{3}{4}x - 5 = 1. \tag{1}$$

Adding 5 to both members,

$$\tfrac{3}{4}x = 6. \tag{2}$$

Dividing by 3, $$\tfrac{1}{4}x = 2. \tag{3}$$

Multiplying by 4, $$x = 8. \tag{4}$$

It is evident that 8 is a root of equations (1), (2), (3), and (4).

In thus applying the principles of Ch. I., Art. 17, we replace the given equation by a simpler one, which has the same root, this equation by a still simpler one, which again has the same root, and so on.

Such equations as (1), (2), (3), and (4) are called **Equivalent Equations.**

In general, *two equations are equivalent when every solution of the first is a solution of the second, and every solution of the second is a solution of the first.*

13. It is important to notice that the use of the principles given in Ch. I., Art. 17, may lead to incorrect results.

Thus, by (iii.), we should be permitted to multiply both members of an equation by an expression which contains the unknown number.

E.g., the equation $x - 3 = 0$ has the root 3.

Multiplying both members by $x - 2$, we obtain

$$(x-3)(x-2) = 0.$$

This equation has the root 3,

since $$(3-3)(3-2) = 0 \cdot 1 = 0;$$

and also the root 2,

since $$(2-3)(2-2) = -1 \cdot 0 = 0.$$

But 2 is not a root of the given equation, since $2-3$ does not equal 0.

That is, *in multiplying both members by $x-2$, we gained a root* 2. Observe that this root is the root of $x-2=0$.

The derived equation is therefore not equivalent to the given one.

Again, by (iii.), we should be permitted to multiply both members of an equation by 0.

Multiplying both members of $x-3=0$, by 0, we have

$$0(x-3)=0.$$

Any number is a root of this equation, since

$$0(1-3)=0,\ 0(2-3)=0,\ 0(3-3)=0,\ 0(4-3)=0, \text{ etc.}$$

Finally, by (iv.), we should be permitted to divide both members of any equation by an expression which contains the unknown number.

E.g., the equation $(x-1)(x+1)=3(x-1)$,

has the root 1, since

$$(1-1)(1+1)=3(1-1), \text{ or } 0\times 2=3\times 0, \text{ or } 0=0;$$

and the root 2, since

$$(2-1)(2+1)=3(2-1), \text{ or } 1\times 3=3\times 1.$$

Dividing both members by $x-1$, we obtain

$$x+1=3.$$

This equation has the root 2 *only*, and not the root 1 of the given equation.

That is, *in dividing both members by $x-1$, we lost the root* 1. Observe that this root is a root of $x-1=0$.

The derived equation is therefore not equivalent to the given one.

14. The correct statements of the principles which are applied in solving equations are, therefore, as follows:

(i.) **Addition and Subtraction.** — *The equation obtained by adding to, or subtracting from, both members of an equation the same number or expression is equivalent to the given one.*

(ii.) **Multiplication and Division.** — *The equation obtained by multiplying or dividing both members of an equation by the same number, not* 0, *or by an expression which does not contain the unknown number or numbers, is equivalent to the given one.*

These principles hold for equations of any degree.

In the solutions of equations in the preceding chapters, we multiplied or divided only by Arabic numerals. Nevertheless, we required each result to be checked.

EXERCISES I.

Solve each of the following equations:

1. $x(x+3)=x(x-5)$. **2.** $3x(x-5)=3x(x+2)$.

3. $2(x+1)-3(x+1)+9(x+1)+18=7(x+1)$.

4. $5(x-7)-4(x-7)+11(x-7)=10+2(x-7)$.

5. $-8(3x-5)+5(3x-5)-17-2(3x-5)=3$.

6. $x(x+1)+x(x+2)=(x+3)(2x-1)$.

7. $(5x-2)(3x-4)=(3x+5)(5x-6)$.

8. $2(x+2)(x+3)=2(x+2)(x-5)$.

9. $(6x-5)(9x-3)+9=6(2-9x)(2-x)$.

10. $(16x+5)(9x+31)=(4x+14)(36x+10)$.

11. $x^2-x[1-x-2(3-x)]=x+1$.

12. $(x+1)(x+1)=[111-(1-x)]x-80$.

13. $2[5(3x+4)+3]+1=77$.

14. $-4-4\{4-4[4-4(4-x)]\}=44$.

15. $3\{3[3(3x+1)+4]+5\}+2=107$.

16. $4\{4[4(4x-3)-3]-3\}-3=1$.

17. $3[5\{5(x-3)-3\}-7]=2(x+2)-3$.

18. $\frac{1}{5}\{\frac{1}{2}[3(x-4)+1]+3\}=1$.

19. $\frac{1}{2}\{\frac{1}{2}[\frac{1}{2}(x+\frac{1}{2})-\frac{1}{2}]+\frac{1}{2}\}=x-2$.

Problems.

Pr. **1.** A man has \$ 4.50 in dimes and dollars, and he has five times as many dimes as dollars. How many coins of each kind has he?

Let x stand for the number of dollars.

Then $5x$ stands for the number of dimes.

We must first express the dimes as fractional parts of dollars, or the dollars as multiples of dimes. The latter method is the simpler. Since one dollar is 10 dimes, x dollars are $10x$ dimes.

The man evidently has 45 dimes.

The problem states,

in *verbal* language: *ten times the number of dollars plus the number of dimes is equal to* 45;

in *algebraic* language: $10x + 5x = 45,$

$$15x = 45;$$

whence $x = 3,$

the number of dollars.

Then $5x, = 15,$ the number of dimes.

Evidently the value of the coins is $3 + \frac{15}{10}$ dollars, or \$ 4.50.

As in this problem, the magnitudes of all concrete quantities of the same kind must be referred to the same unit; if x stand for a certain number of yards, then all other distances must likewise stand for numbers of yards, not of miles or of feet.

Pr. **2.** I have in mind a number of six digits, the last one on the left being 1. If I bring this digit to the first place on the right, I shall obtain a number which is three times the number I have in mind. What is the number?

Let x stand for the number which is composed of the five digits on the right of 1.

Then the original number is $100{,}000 + x$.

When 1 is moved to the first place on the right, each digit in x is moved one place to the left. Therefore, the resulting number is $10x + 1$.

The problem states,

in *verbal* language: *the resulting number is equal to three times the original number;*

in *algebraic* language: $10x + 1 = 3(100{,}000 + x)$,

whence $7x = 299{,}999$,

and $x = 42{,}857$.

Therefore the required number is 142,857.

Pr. 3. A man asked another what time it was, and received the answer: "It is between 5 and 6 o'clock, and the minute-hand is directly over the hour-hand." What time was it?

At 5 o'clock, the minute-hand points to 12 and the hour-hand to 5. The hour-hand is therefore 25 minute-divisions in advance of the minute-hand.

Let x stand for the number of minute-divisions passed over by the minute-hand from 5 o'clock until it is directly over the hour-hand between 5 and 6 o'clock.

Since the minute-hand must pass over 25 more minute-divisions than the hour-hand in order to overtake the latter, the number of minute-divisions passed over by the hour-hand is $x - 25$.

The problem states, or implies,

in *verbal* language: *the number of minute-divisions passed over by the minute-hand is* 12 *times the number of minute-divisions passed over by the hour-hand;*

in *algebraic* language: $x = 12(x - 25)$.

From this equation we obtain $x = 27\frac{3}{11}$. Consequently, the two hands coincide at $27\frac{3}{11}$ minutes past 5 o'clock.

EXERCISES II.

1. The sum of three consecutive numbers exceeds the second by 42. What are the numbers?

2. A and B divide a sum of money. A receives \$3 as often as B receives \$5. If A receives \$$3x$, how many dollars does B receive?

3. A and B divide \$1200. A receives \$3 as often as B receives \$5. How many dollars does each receive?

4. The length of a room is four times its width. If it were 12 feet shorter and 12 feet wider, it would be square. What are the dimensions of the room?

5. A man travels 144 miles by train, boat, and stage. He travels 20 miles farther by boat than by stage, and three times as far by train as by boat and stage together. How many miles does he travel by each conveyance?

6. A man paid a debt in four monthly payments. He paid \$45 more each month than the preceding. If his debt was three times his last payment, how much was his first payment? How much was his debt?

7. In a number of two digits, the tens' digit is three times the units' digit. The number itself exceeds four times the units' digit by 54. What is the number?

8. In a number of two digits, the tens' digit is twice the units' digit. If the digits are interchanged, twice the resulting number exceeds the original number by 9. What is the number?

9. Three boys, A, B, and C, have a number of marbles. A and B have 55, B and C have 62, and A and C have 57. How many marbles has each boy?

10. A man, wishing to give alms to several beggars, lacks 15 cents of enough to give 22 cents to each one. If he were to give 20 cents to each one, he would have 1 cent left over. How many beggars are there?

11. A, travelling 25 miles a day, has 3 days' start of B, who travels 30 miles a day in the same direction. After how many days will B overtake A?

12. The sum of two numbers is 47, and their difference increased by 7 is equal to the less. What are the numbers?

13. The sum of three consecutive even numbers exceeds the least by 42. What are the numbers?

14. Atmospheric air is a mixture of four parts of nitrogen with one of oxygen. How many cubic feet of oxygen are there in a room 12 yards long, 5 yards wide, and 17 feet high?

15. A merchant paid $7.50 in an equal number of dimes and five-cent pieces. How many coins of each kind did he pay?

16. A man has $5.70 in dimes and quarters, and he has 6 more quarters than dimes. How many coins of each kind has he?

17. In my right pocket I have as many dollars as I have cents in my left pocket. If I transfer $6.93 from my right pocket to my left, I shall have as many dollars in my left pocket as I shall have cents in my right. How much money have I in my left pocket?

18. One barrel contained 36 gallons, and another 60 quarts, of wine. From the first three times as much wine was drawn as from the second; the first then contained twice as much wine as the second. How much wine was drawn from each?

19. A regiment moves from A to B, marching 18 miles a day. Two days later a second regiment leaves B for A, and marches 26 miles a day. At what distance from A do the regiments meet, A being 212 miles from B?

20. A man travels 3 miles in one hour. During the first half-hour, he goes 10 yards farther every minute than during the second half-hour. How many yards a minute does he go the first half-hour?

21. The greatest of three vessels holds 28 gallons more than the second, and 45 gallons more than the third. If the contents of the second and third, when full, are poured into the first, when empty, the latter will lack 8 gallons of being filled. What is the capacity of each vessel?

22. A father leaves $ 25,800 to his four sons. The first receives twice as much as the second, less $ 300; the second three times as much as the third, less $ 600; and the third four times as much as the fourth, less $ 900. How many dollars does each son receive?

23. Two bodies move from the same point in the same direction, one at the rate of 24 feet a minute, the other at the rate of 30 feet a minute. If the second starts 35 minutes after the first, where will it overtake the first? When will the distance between them be 270 feet before they meet? When 270 feet after they meet?

24. A child was born in November. On the 10th of December the number of days in its age was equal to the number of days from the 1st of November to the day of its birth, inclusive. What was the date of its birth?

25. A person attempts to arrange a number of coins in the form of a square. On the first attempt, he has 31 pieces left over. When he adds 2 to each side of his square, he lacks 25 coins of enough to complete this square. How many coins has he?

26. In a certain family each son has as many brothers as sisters, but each daughter has twice as many brothers as sisters. How many children are in the family?

27. A merchant's investment yields him yearly $33\frac{1}{3}\%$ profit. At the end of each year, after deducting $ 1000 for personal expenses, he adds the balance of his profits to his invested capital. At the end of three years his capital is twice his original investment. How much did he invest?

28. I have in mind a number of four digits, the first one on the right being 2. If I bring this digit to the last place on the left, I shall obtain a number which is less than the number I have in mind by 2106. What is the number?

29. At what time between 3 and 4 o'clock will the minute-hand of a watch be directly over the hour-hand? At what time between 9 and 10 o'clock?

CHAPTER V.

TYPE-FORMS.

1. We shall in this chapter consider a number of products and quotients which are of frequent occurrence. They enable us to shorten work by writing similar products and quotients without performing the actual multiplications and divisions. They are called **Type-Forms.**

TYPE-FORMS IN MULTIPLICATION.

The Square of a Binomial.

2. By actual multiplication, we have

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ba+b^2=a^2+2\,ab+b^2.$$

That is, *the square of the sum of two numbers is equal to the square of the first number, plus twice the product of the two numbers, plus the square of the second number.*

$$\begin{aligned} \textit{E.g.,}\qquad (2\,x+5\,y)^2 &= (2\,x)^2+2\,(2\,x)(5\,y)+(5\,y)^2 \\ &= 4\,x^2+20\,xy+25\,y^2. \end{aligned}$$

3. By actual multiplication, we have

$$(a-b)^2=(a-b)(a-b)=a^2-ab-ba+b^2=a^2-2\,ab+b^2.$$

That is, *the square of the difference of two numbers is equal to the square of the first number, minus twice the product of the two numbers, plus the square of the second number.*

$$\begin{aligned} \textit{E.g.,}\qquad (3\,x-7\,y)^2 &= (3\,x)^2-2\,(3\,x)(7\,y)+(7\,y)^2 \\ &= 9\,x^2-42\,xy+49\,y^2. \end{aligned}$$

4. Observe that this type-form is equivalent to that of Art. 2, since $a - b = a + (-b)$.

E.g., $$(3x - 7y)^2 = (3x)^2 + 2(3x)(-7y) + (-7y)^2$$
$$= 9x^2 - 42xy + 49y^2, \text{ as above.}$$

The signs of all the terms of an expression which is to be squared may be changed without changing the result.

For, $$(a - b)^2 = [-(b - a)]^2 = (b - a)^2$$

5. In applying the type-forms in this Chapter, it will be necessary to raise a monomial to any required power.

We have

$$(5a^3b^4)^2 = 5 \cdot 5a^3a^3b^4b^4 = 5^2a^{3+3}b^{4+4} = 5^2a^{2\times 3}b^{2\times 4} = 25a^6b^8.$$

That is, to square a monomial:

Square the numerical coefficient, and multiply the exponent of each literal factor by 2.

In general, to raise a given monomial to any required power:

Raise the numerical coefficient to the required power, and multiply the exponent of each literal factor by the exponent of the required power.

E.g., $$(3ab^2)^3 = 3^3a^3b^{2\times 3} = 27a^3b^6.$$

EXERCISES I.

Write, without performing the actual multiplications, the values of:

1. $(x+1)^2$. **2.** $(x-3)^2$. **3.** $(a+5)^2$.

4. $(x-4)^2$. **5.** $(3x+2)^2$. **6.** $(4-5z)^2$.

7. $(mn+6)^2$. **8.** $(ab-8)^2$. **9.** $(xy+z)^2$.

10. $(4x^2-3)^2$. **11.** $(3xy+5z)^2$. **12.** $(2ab-6bc)^2$.

13. $(xy^2-3x^2y)^2$. **14.** $(2a^2b^2-9c^2)^2$. **15.** $(4a^2b^3-8c^4)^2$.

16. $(x^n+1)^2$. **17.** $(x^m-y^n)^2$. **18.** $(a^{n+1}+a^{n-1})^2$.

Simplify the following expressions:

19. $a^2+b^2-(a-b)^2$. **20.** $(x-y)^2-(x+y)^2$.

21. $x^2+y^2-4x+6y+3$, when $x=a+1$, $y=a-2$.

22. $(a+b-c)(a+b)+(a-b+c)(a+c)+(b+c-a)(b+c)$.

Verify the following identities:

23. $(a^2 + b^2)(x^2 + y^2) - (ax + by)^2 = (ay - bx)^2$.

24. $a^2 + b^2 + 4c^2 + 2ab + 8bc = 4(a + c)^2$, when $b = a$.

Product of the Sum and Difference of Two Numbers.

6. By actual multiplication, we have

$$(a + b)(a - b) = a^2 - ab + ba - b^2 = a^2 - b^2.$$

That is, *the product of the sum of two numbers and the difference of the same numbers, taken in the same order, is equal to the square of the first, minus the square of the second.*

Ex. 1. $(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2 = 4x^2 - 9y^2$.

The product of two multinomials can frequently be brought under this type-form by properly grouping terms.

Ex. 2.
$$\begin{aligned}(x^2 + x + 1)(x^2 - x + 1) &= [(x^2 + 1) + x][(x^2 + 1) - x]\\ &= (x^2 + 1)^2 - x^2\\ &= x^4 + 2x^2 + 1 - x^2\\ &= x^4 + x^2 + 1.\end{aligned}$$

Ex. 3.
$$\begin{aligned}(x - y + z)(x + y - z) &= [x - (y - z)][x + (y - z)]\\ &= x^2 - (y - z)^2\\ &= x^2 - (y^2 - 2yz + z^2)\\ &= x^2 - y^2 - z^2 + 2yz.\end{aligned}$$

EXERCISES II.

Write, without performing the actual multiplications, the values of:

1. $(a + 2)(a - 2)$.
2. $(x - 6)(x + 6)$.
3. $(m + 9)(m - 9)$.
4. $(2a + 1)(2a - 1)$.
5. $(5x - 7)(5x + 7)$.
6. $(9 - 5x)(9 + 5x)$.
7. $(2a + 3b)(2a - 3b)$.
8. $(5x - 6y)(5x + 6y)$.
9. $(-8m + 5n)(8m + 5n)$.
10. $(ab + 1)(ab - 1)$.
11. $(3ax - 4)(3ax + 4)$.
12. $(-xy + z)(xy + z)$.
13. $(-2ab + c)(2ab + c)$.
14. $(5xy - 3z)(5xy + 3z)$.

15. $(x^2+1)(x^2-1)$. **16.** $(3a^3+4)(3a^3-4)$.

17. $(5a^4-2b)(5a^4+2b)$. **18.** $(3x^3y^2-5z^2)(3x^3y^2+5z^2)$.

19. $(3a^n+5)(3a^n-5)$.

20. $(-5x^{n+1}+9x^{n-1})(5x^{n+1}+9x^{n-1})$.

21. $[a^2+6(a+b)][a^2-6(a+b)]$.

22. $(x+y+5)(x+y-5)$.

23. $(4a-3b-7)(4a-3b+7)$.

24. $(x^2+y^2+z^2)(-x^2+y^2+z^2)$.

25. $(a^2-ab+b^2)(a^2+ab+b^2)$.

26. $(x^2+2x-1)(x^2-2x-1)$.

27. $(x^4-x^2+1)(x^4+x^2-1)$.

28. $(-a^2-b^2+3)(a^2-b^2+3)$.

Simplify the following expressions:

29. $(1+x)^2-(1-x)(1+x)$.

30. $(2x+3y)^2(2x-3y)^2$.

31. $(x-3)(x-1)(x+1)(x+3)$.

32. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)$.

33. $(x^2-1)(x^8+1)(x^4+1)(x^2+1)$.

34. $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$.

35. $(a+b-c)(a+c-b)(b+c-a)(a+b+c)$.

The Product $(x+a)(x+b)$.

7. By actual multiplication, we have

$$(x+a)(x+b)=x^2+ax+bx+ab=x^2+(a+b)x+ab;$$
$$(x+a)(x-b)=x^2+ax-bx-ab=x^2+(a-b)x-ab;$$
$$(x-a)(x-b)=x^2-ax-bx+ab=x^2-(a+b)x+ab.$$

We thus derive the following method for multiplying two binomials which have a common first term:

The first term of the product is the square of the common first terms of the binomials.

The coefficient of the second term of the product is the algebraic sum of the second terms of the binomials.

The last term of the product is the product of the last terms of the binomials.

Ex. 1. Write the product $(x+3)(x+7)$.

The first term is x^2;

The second term is $(3+7)x, \; = 10x$;

The third term is $\quad 3 \times 7 = 21$.

Therefore $\quad (x+3)(x+7) = x^2 + 10x + 21$.

Ex. 2. Write the product $(x-8)(x+2)$.

First term: x^2; second term: $(-8+2)x, \; = -6x$;

third term: $-8 \times 2 = -16$.

Therefore $\quad (x-8)(x+2) = x^2 - 6x - 16$.

Ex. 3. Write the product $(a^2+9)(a^2-3)$.

First term: $(a^2)^2, \; = a^4$; second term: $(9-3)a^2, \; = 6a^2$;

third term: $9 \times (-3), \; = -27$.

Therefore $\quad (a^2+9)(a^2-3) = a^4 + 6a^2 - 27$.

Ex. 4. Write the product $(x-5y)(x-7y)$.

First term: x^2; second term: $(-5y-7y)x, \; = -12xy$;

third term: $-5y \times (-7y), \; = 35y^2$.

Therefore $(x-5y)(x-7y) = x^2 - 12xy + 35y^2$.

EXERCISES III.

Write, without performing the actual multiplications, the values of:

1. $(x+2)(x+3)$.
2. $(x+2)(x-3)$.
3. $(x-2)(x+3)$.
4. $(x-2)(x-3)$.
5. $(x+5)(x+8)$.
6. $(x+5)(x-8)$.
7. $(x-5)(x+8)$.
8. $(x-5)(x-8)$.
9. $(8+m)(m-9)$.
10. $(5+a)(a-6)$.
11. $(7+3x)(7-x)$.
12. $(-3+5a)(6+5a)$.
13. $(x+y)(x+2y)$.
14. $(x+y)(x-2y)$.

15. $(x-y)(x-2y)$.
16. $(ab+1)(ab-3)$.
17. $(xy+7)(xy-8)$.
18. $(ab+3c)(ab-5c)$.
19. $(x^2+8)(x^2-9)$.
20. $(x^2y-5)(x^2y+11)$.
21. $(xy^2+9a)(xy^2-6a)$.
22. $(x^2+3ab)(x^2-2ab)$.
23. $(a^n+2)(a^n-5)$.
24. $(x^{m-1}-3)(x^{m+1}+8)$.
25. $(a+b+3)(a+b-7)$.
26. $(x-y+3z)(x-y-5z)$.

The Product $(ax+b)(cx+d)$.

8. By actual multiplication, we obtain

$$(ax+b)(cx+d)=acx^2+(ad+bc)x+bd.$$

In this type-form that part of the multiplication which gives the middle term of the type-form may be represented concisely by the following arrangement:

$$\begin{array}{c} cx+d \\ \times \\ \underline{ax+b} \\ (ad+bc)x \end{array}$$

The products of the terms connected by the cross lines are called *cross-products,* and their sum is the middle term of the given trinomial.

That is, *the product of two binomials, arranged to powers of a common letter, is equal to the product of the first terms, plus the sum of the cross-products, plus the product of the last terms.*

Ex. 1. $(7x-5y)(2x+3y)=7x\cdot 2x+(7\cdot 3-5\cdot 2)xy-5y\cdot 3y$
$$=14x^2+11xy-15y^2.$$

EXERCISES IV.

Write, without performing the actual multiplications, the values of:

1. $(3a+1)(5a+2)$.
2. $(7x-3)(3x-1)$.
3. $(5x+7)(3x-2)$.
4. $(2x-9)(5x+1)$.
5. $(2x+15)(4x-5)$.
6. $(11a-3)(9a+7)$.
7. $(2a+b)(3a-b)$.
8. $(2a-b)(3a+b)$.

9. $(3x - y)(2x - y)$.
10. $(7a + 3b)(5a + 2b)$.
11. $(6x - 7y)(3x + 2y)$.
12. $(5x - 3z)(2x + 5z)$.
13. $(7y + 2u)(8y - 7u)$.
14. $(2ab - x)(3ab + x)$.
15. $(5mn + 3p)(6mn + 7p)$.
16. $(9m^2 - 3)(8m^2 + 11)$.
17. $(3x^2 + 5y^2)(2x^2 - 3y^2)$.
18. $[3(a + b) + 5][5(a + b) - 2]$.
19. $[2(x - y) + 7][3(x - y) + 2]$.

TYPE-FORMS IN DIVISION.

Quotient of the Sum or the Difference of Like Powers of two Numbers by the Sum or the Difference of the Numbers.

9. By actual division, we obtain

$$(a^2 - b^2) \div (a + b) = a - b \text{ and } (a^2 - b^2) \div (a - b) = a + b.$$

That is, *the difference of the squares of two numbers is divisible by the sum, and also by the difference of the numbers. The quotient in the first case is the difference of the numbers, taken in the same order, and in the second case is the sum of the numbers.*

Ex. 1. $(9 - 25x^2) \div (3 + 5x) = 3 - 5x$.

Ex. 2. $(16x^4 - 81y^6) \div (4x^2 - 9y^3) = 4x^2 + 9y^3$.

EXERCISES V.

Write, without performing the actual divisions, the values of:

1. $(x^2 - 1) \div (x - 1)$.
2. $(25 - x^2) \div (5 + x)$.
3. $(4a^2 - 9) \div (2a - 3)$.
4. $(\frac{1}{9} - x^2y^2) \div (\frac{1}{3} + xy)$.
5. $(x^4 - 1) \div (x^2 + 1)$.
6. $(4a^4 - b^2) \div (2a^2 - b)$.
7. $(16x^2 - 9y^2) \div (4x - 3y)$.
8. $(64a^2b^2 - 121c^2) \div (8ab + 11c)$.
9. $(4a^4x^6 - y^8) \div (2a^2x^3 + y^4)$.
10. $(25a^{10} - 16x^6y^2) \div (5a^5 - 4x^3y)$.
11. $(x^{2n} - 1) \div (x^n - 1)$.
12. $(a^{4n} - 16b^{16}) \div (a^{2n} + 4b^8)$.
13. $(x^{2n+2} - 4) \div (x^{n+1} + 2)$.
14. $(a^{8n} - b^{4n+4}) \div (a^{4n} - b^{2n+2})$.
15. $[(a + b)^2 - 1] \div (a + b + 1)$.
16. $[4 - (a + b)^2] \div (2 - a - b)$.

17. $(a^2 - 2ab + b^2 - 1) \div (a - b + 1)$.

18. $(a^2 - n^2 - p^2 + 2np) \div (a - n + p)$.

19. $(p^2 - r^2 - 4 - 4r) \div (p - r - 2)$.

20. $[(a^2 + 2ab + b^2)x^6 - y^4] \div [(a + b)x^3 + y^2]$.

21. $(x^4 + 2x^2y^2 + y^4 - z^2 - 2zu - u^2) \div (x^2 + y^2 + u + z)$.

22. $(a^2 - b^2 + 2bz - 2ax + x^2 - z^2) \div (a - x - b + z)$.

The Sum and Difference of Two Cubes

10. By actual division, we obtain

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2. \qquad (1)$$

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2. \qquad (2)$$

That is, *the sum of the cubes of two numbers is divisible by the sum of the numbers. The quotient is the square of the first number, minus the product of the numbers, plus the square of the second number.*

The principle contained in (2) may be stated in a similar way.

Ex. 1. $(8x^3 + \frac{1}{125}) \div (2x + \frac{1}{5}) = (2x)^2 - (2x)(\frac{1}{5}) + (\frac{1}{5})^2$
$= 4x^2 - \frac{2}{5}x + \frac{1}{25}$.

Ex. 2. $(a^{12} - b^9) \div (a^4 - b^3) = (a^4)^2 + a^4b^3 + (b^3)^2$
$= a^8 + a^4b^3 + b^6$.

EXERCISES VI.

Write, without performing the actual divisions, the values of:

1. $(1 + a^3) \div (1 + a)$.

2. $(x^3 - 8) \div (x - 2)$.

3. $(m^3 + 27) \div (m + 3)$.

4. $(64 - x^3) \div (4 - x)$.

5. $(216 + a^3) \div (6 + a)$.

6. $(8a^3 - 27) \div (2a - 3)$.

7. $(x^3y^3 + 1) \div (xy + 1)$.

8. $(8a^3b^6 + 27) \div (2ab^2 + 3)$.

9. $(125x^3y^9 - z^6) \div (5xy^3 - z^2)$.

10. $(27a^6b^9 - 64c^3) \div (3a^2b^3 - 4c)$.

11. $(8m^{15}n^3 - p^{12}) \div (2m^5n - p^4)$.

12. $(a^{3n}+1)\div(a^n+1)$. **13.** $(x^{6m}-y^{3n})\div(x^{2m}-y^n)$.

14. $(343\,x^{3n-3}+y^{6n})\div(7\,x^{n-1}+y^{2n})$.

15. $[(x+y)^3-8]\div(x+y-2)$.

16. $[1+(x-y)^3]\div(1+x-y)$.

17. $[(a-b)^6-8\,c^3]\div[a^2+b^2-2(ab+c)]$.

Sum and Difference of Like Powers of Two Numbers.

11. By actual division, we find:

$$(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3;$$

$$(a^4-b^4)\div(a-b)=a^3+a^2b+ab^2+b^3;$$

a^4+b^4 **is not divisible by either** $a+b$ **or** $a-b$;

$$(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4;$$

a^5+b^5 **is not divisible by** $a-b$.

The above identities and those of Arts. 9–10, illustrate the following principles:

(i.) *a^n-b^n is divisible by $a-b$, but not by $a+b$, when n is odd.*

The quotient is

$$a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+a^2b^{n-3}+ab^{n-2}+b^{n-1}.$$

(ii.) *a^n-b^n is divisible by both $a+b$ and $a-b$, when n is even.*

The quotient, when $a+b$ is the divisor, is

$$a^{n-1}-a^{n-2}b+a^{n-3}b^2-\cdots-a^2b^{n-3}+ab^{n-2}-b^{n-1};$$

and, when $a-b$ is the divisor, is

$$a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+a^2b^{n-3}+ab^{n-2}+b^{n-1}.$$

(iii.) *a^n+b^n is divisible by $a+b$, but not by $a-b$, when n is odd.*

The quotient is

$$a^{n-1}-a^{n-2}b+a^{n-3}b^2-\cdots+a^2b^{n-3}-ab^{n-2}+b^{n-1}.$$

(iv.) *a^n+b^n is not divisible by either $a+b$ or $a-b$, when n is even.*

12. The following directions will be helpful in writing the quotients of these type-forms:

(i.) *When the divisor is a sum, the signs of the terms of the quotient alternate,* $+$ *and* $-$.

(ii.) *When the divisor is a difference, the signs of the terms of the quotient are all* $+$.

(iii.) *In the first term of the quotient the exponent of* a *is less by* 1 *than its exponent in the dividend, and decreases by* 1 *from term to term.*

(iv.) *The exponent of* b *is* 1 *in the second term of the quotient, and increases by* 1 *from term to term.*

Observe that the quotient is homogeneous in a and b, of degree less by 1 than the degree of the dividend.

Ex. 1. $(x^4 - 16y^4) \div (x - 2y)$

$$= [x^4 - (2y)^4] \div (x - 2y)$$
$$= x^3 + x^2(2y) + x(2y)^2 + (2y)^3$$
$$= x^3 + 2x^2y + 4xy^2 + 8y^3.$$

Ex. 2. $(x^5 - 32) \div (x - 2) = x^4 + 2x^3 + 4x^2 + 8x + 16.$

EXERCISES VII.

Write, without performing the actual divisions, the values of:

1. $(x^4 - 1) \div (x + 1)$.
2. $(1 - a^4) \div (1 - a)$.
3. $(m^5 - 1) \div (m - 1)$.
4. $(32 + n^5) \div (2 + n)$.
5. $(a^6 - b^6) \div (a - b)$.
6. $(a^7 + b^7) \div (a + b)$.
7. $(a^{10} - b^{10}) \div (a - b)$.
8. $(a^{11} + b^{11}) \div (a + b)$.
9. $(x^8 - y^8) \div (x^2 - y^2)$.
10. $(x^{10} + y^{10}) \div (x^2 + y^2)$.
11. $(a^8y^4 - b^{12}) \div (a^2y + b^3)$.
12. $(x^{10}y^5 - z^{15}) \div (x^2y - z^3)$.
13. $(x^6 - 64y^{12}) \div (x - 2y^2)$.
14. $(243a^5b^{15} + 32c^{10}) \div (3ab^3 + 2c^2)$.
15. $(x^{4n} - y^{4n}) \div (x - y)$.
16. $(x^{5n} - 1) \div (x^n - 1)$.
17. $(1 + a^{5m}) \div (1 + a)$.
18. $(a^{14}x^{7n} + b^{14m}) \div (a^2x^n + b^{2m})$.

CHAPTER VI.

FACTORS AND MULTIPLES OF INTEGRAL ALGEBRAIC EXPRESSIONS.

INTEGRAL ALGEBRAIC FACTORS.

1. A product of two or more factors is, by the definition of division, exactly divisible by any one of them.

An **Integral Algebraic Factor** of an expression is an integral expression which exactly divides the given one.

E.g., integral factors of $6a^2x$ are 6, a^2x, $3x$, $2a^2$, etc.;

integral factors of $a^2 - b^2$ are $a + b$ and $a - b$.

2. A **Prime Factor** is one which is exactly divisible only by itself and unity.

E.g., the prime factors of $6a^2x$ are 2, 3, a, a, x.

A **Composite Factor** is one which is not prime, *i.e.*, which is itself the product of two or more prime factors.

E.g., composite factors of $6a^2x$ are 6, ax, $2a$, $3ax$, etc.

3. Any monomial can be resolved into its prime factors by inspection.

E.g., the prime factors of $4a^3b^2$ are 2, 2, a, a, a, b, b.

Multinomials whose Terms have a Common Factor.

4. From Ch. III., Art. 30, we have

$$ab + ac - ad = a(b+c-d). \qquad (1)$$

This relation may be called the *Fundamental Formula for Factoring.* From it we derive the following method for find-

ing the second factor of a multinomial whose terms have a common factor:

Determine by inspection the remaining factors of its terms, and take their algebraic sum.

5. Ex. 1. Factor $2x^2y - 2xy^2$.

The factor $2xy$ is common to both terms; the remaining factor of the first term is x, that of the second term is $-y$, and their algebraic sum is $x - y$.

Consequently, $2x^2y - 2xy^2 = 2xy(x - y)$.

Ex. 2. $ab^2 + abc + b^2c = b(ab + ac + bc)$.

6. In the fundamental formula the letters a, b, c, d may stand for binomial or multinomial expressions.

Ex. 1. Factor $a(x - 2y) + b(x - 2y)$.

The factor $x - 2y$ is common to both terms; the remaining factor of the first term is a, that of the second term is b, and their algebraic sum is $a + b$.

Consequently $a(x - 2y) + b(x - 2y) = (x - 2y)(a + b)$.

Ex. 2.
$$\begin{aligned} x^2(1 - m) - y^2(m - 1) &= x^2(1 - m) + y^2(1 - m) \\ &= (1 - m)(x^2 + y^2). \end{aligned}$$

EXERCISES I.

Factor the following expressions:

1. $5x + 5$.
2. $ax - a$.
3. $4a^3 - 6$.
4. $x^4 - 2x^3$.
5. $a^2b + ab^2$.
6. $2an - 4n^2$.
7. $3x^3y^2 - 5x^2y^3$.
8. $12a^3b^3 + 3a^2b^2$.
9. $10a^4x^2 - 15a^2x^4$.
10. $3ab + 6ac - 12ad$.
11. $70xy - 98y^2 - 140yz$.
12. $\frac{15}{16}ax + \frac{15}{16}bx^2 + \frac{1}{4}x$.
13. $6ax^4 - 15a^3bx^5 + 18a^2b^2x^6$.
14. $8a^2n^5x^5 - 10an^4x^7 + 4a^2n^3x^8$.
15. $45m^3n^3p + 90m^3n^2p - 75m^2np^2$.
16. $28a^5b^3c - 84a^3b^4c^2 + 98a^4b^4c^3$.
17. $27x^3y^4z^2 + 135x^5y^4z^4 - 81x^4y^4z^4$.

18. $x-(n+1)x$.
19. $a^2(a+x)+x^2(a+x)$.
20. $3a(a-1)-3(a-1)$.
21. $2(n+1)^2-4(n+1)$.
22. $a(x-1)-x+1$.
23. $m(q-p)-(p-q)$.
24. $6m^{n+1}-3m^{n+2}+9m^{n+3}$.
25. $a^{n+1}-a+a^{n-1}$.
26. $5^{n+3}-125x+625x^2$.
27. $2^{n+4}-8\times 2^{n-1}+16$.

Grouping Terms.

7. When all the terms of a given expression do not contain a common factor, it is sometimes possible to group the terms so that all the groups shall contain a common factor.

Ex. 1. Factor $2a+2b+ax+bx$.

Factoring the first two terms by themselves, and the last two terms by themselves, we obtain

$$2(a+b)+x(a+b)=(a+b)(2+x).$$

Ex. 2. $$\begin{aligned} x^2-xy-xz+yz &= (x^2-xy)-(xz-yz) \\ &= x(x-y)-z(x-y)=(x-y)(x-z). \end{aligned}$$

Ex. 3. $$\begin{aligned} x^3+3x^2-2x-6 &= (x^3+3x^2)-(2x+6) \\ &= x^2(x+3)-2(x+3) \\ &= (x+3)(x^2-2). \end{aligned}$$

EXERCISES II.

Factor the following expressions:

1. $am+an+bm+bn$.
2. $ax-by-bx+ay$.
3. $m^2-am+bm-ab$.
4. $x^2-5x-2xy+10y$.
5. $ax+a+x+1$.
6. $na-a+n-1$.
7. $mz+m-z-1$.
8. x^3-x^2+x-1.
9. $x-y-xy+1$.
10. $1-3a-b+3ab$.
11. $a^3-a^2c+ac^2-c^3$.
12. $3x^4-x^3+6x-2$.
13. $3c^4-3c^3n+cn^2-n^3$.
14. $5ax-cx-5ay+cy$.
15. $2ax-3by-2ay+3bx$.
16. $ac-5ad+3bc-15bd$.
17. $3n^3+nx^2-6n^2x-2x^3$.
18. $18n^2x-12x-9n^2+6$.

19. $18\,ax + 30\,ay - 9\,bx - 15\,by.$

20. $20\,ad - 35\,bd - 8\,ax + 14\,bx.$

21. $24\,mn - 44\,n^2 - 30\,mx + 55\,nx.$

22. $12\,a^3b^4 - 4\,a^2b^4 - 4\,a^2b^3 + 12\,a^3b^3.$

23. $a^4 - a^3n^2 + a^2n - an^3 + n^5 - an^3.$

24. $x^4 - ax^3 + 3\,a^2x^2 - 2\,a^2bx^2 + 2\,a^3bx - 6\,a^4b.$

25. $ax + by + cz + bx + cy + az + cx + ay + bz.$

26. $ax - by + cz - bx - cy - az - cx + ay + bz.$

27. $ax + by + cz - bx - cy + az + cx - ay - bz.$

28. $ax + by + cz - bx + cy - az - cx - ay + bz.$

29. $x^3 + 4\,x^2 - 3\,x - 12.$ **30.** $x^3 - 3\,x^2 + 5\,x - 15.$

31. $x^3 + 2\,x^2 + 8\,x + 16.$ **32.** $x^3 - 7\,x^2 - 4\,x + 28.$

Use of Type-Forms in Factoring.

8. If an expression is in the form of one of the type-forms considered in Ch.V., or if it can be reduced to such a form, its factors can be written by inspection.

Trinomial Type-Forms.

9. From Ch. V., Arts. 2 and 3, we have

$$a^2 + 2\,ab + b^2 = (a + b)^2,$$

$$a^2 - 2\,ab + b^2 = (a - b)^2.$$

Therefore a trinomial which is the square of a binomial must satisfy the following conditions:

(i.) *One term of the trinomial is the square of the first term of the binomial.*

(ii.) *A second term of the trinomial is the square of the second term of the binomial.*

(iii.) *The remaining term of the trinomial is twice the product of the two terms of the binomial.*

10. Ex. 1. Factor $x^2 + 6x + 9$.

x^2 is the square of x, 9 is the square of 3, and $6x = 2 \cdot x \cdot 3$.

Therefore $$x^2 + 6x + 9 = (x + 3)^2.$$

Ex. 2. Factor $-4xy + 4x^2 + y^2$.

$4x^2$ is the square of $2x$, or of $-2x$; y^2 is the square of y, or of $-y$. Since the term $-4xy$ is negative, one term of the binomial is negative, the other positive.

Therefore $-4xy + 4x^2 + y^2 = (2x - y)^2 = (-2x + y)^2$.

EXERCISES III.

Factor the following expressions:

1. $x^2 - 2x + 1$.
2. $a^2 + 6a + 9$.
3. $y^2 + 12y + 36$.
4. $a^2 - 10ab + 25b^2$.
5. $4x^2 - 12xy + 9y^2$.
6. $9a^2 + 30a + 25$.
7. $20x - 4x^2 - 25$.
8. $36x - 4x^2 - 81$.
9. $16a^2 + 40ab + 25b^2$.
10. $49x^2 - 28xy + 4y^2$.
11. $a^4 - 2a^2x + x^2$.
12. $x^4 - 2x^2y^2 + y^4$.
13. $4ax + 2a^2 + 2x^2$.
14. $6a^2x^2 - 3a^2x^3 - 3a^2x$.
15. $a^2x^2 - 4ac^3x + 4c^6$.
16. $9x^2y^2 - 30xyz^2 + 25z^4$.
17. $24xy - 9x^2 - 16y^2$.
18. $2a^2x^2 - a^4 - x^4$.
19. $4x^{2n} - 12x^n + 9$.
20. $36a^{n+2} - 48a^n + 16a^{n-2}$.
21. $4a^4b^2 - 12a^2bc^2 + 9c^4$.
22. $25m^4n^4 - 60m^2n^2p^2 + 36p^4$
23. $16x^6y^4 - 24x^3y^2z^3 + 9z^6$.
24. $49a^4b^6 + 70a^2b^3c^4 + 25c^8$.
25. $(a + x)^2 + 2(a + x) + 1$.
26. $(2x - 9)^2 - 6(9 - 2x) + 9$.
27. $xy - xz - (y^2 - 2yz + z^2)$.
28. $a^2 + 2an + n^2 - ap - pn$.
29. $2a + ad - d^2 - 4d - 4$.
30. $a^2 + 2ab - 4ac - 4bc + 4c^2$.
31. $x^2 - 6yz - 4xy + 3xz + 4y^2$.
32. $a^4b^4 + 2a^3b^3 + 2a^2b^2 + 2ab + 1$.

11. From Ch. V., Art. 7, we have

$$x^2 + (a + b)x + ab = (x + a)(x + b).$$

When a trinomial, arranged to descending powers of some letter, say x, can be factored into two binomials, it must satisfy the following conditions:

(i.) *One term of the trinomial is the square of the letter of arrangement,* i.e., *of the common first term of the binomial factors.*

(ii.) *The coefficient of the first power of the letter of arrangement in the trinomial is the algebraic sum of two numbers whose product is the remaining term of the trinomial.*

(iii.) *These two numbers are the second terms of the binomial factors.*

12. Ex. 1. Factor $x^2 + 8x + 15$.

The common first term of the binomial factors is evidently x. The second terms are two numbers whose product is 15, and whose sum is 8. By inspection we see that

$$3 + 5 = 8 \text{ and } 3 \times 5 = 15;$$

that is, the second terms of the binomial factors are 3 and 5.

Consequently, $x^2 + 8x + 15 = (x + 3)(x + 5)$.

Ex. 2. Factor $x^2 - 7x + 12$.

The common first term of the binomial factors is x. The second terms are two numbers whose product is 12, and whose sum is -7. Since their product is *positive*, they must be *both positive* or *both negative;* and since their sum is negative, they must be *both negative.*

The possible pairs of negative factors of 12 are: -1 and -12; -2 and -6; -3 and -4.

But since $$-3 + (-4) = -7,$$

the second terms of the binomial factors are -3 and -4.

Consequently, $x^2 - 7x + 12 = (x - 3)(x - 4)$.

Ex. 3. Factor $a^2x^2 + 5ax - 24$.

The common first term of the binomial factors is ax. The second terms are two numbers whose product is -24, and whose sum is 5. Since their product is negative, one must be positive and the other negative; and since their sum is positive, the positive number must have the greater absolute value. The possible pairs of factors of -24 are: -1 and 24; -2 and 12; -3 and 8; -4 and 6.

But since $$-3 + 8 = 5,$$

the second terms of the binomial factors are -3 and 8.

Consequently $a^2x^2 + 5ax - 24 = (ax - 3)(ax + 8)$.

Ex. 4. Factor $x^2 - 3xy - 28y^2$.

The common first term of the binomial factors is x. The second terms are two numbers whose product is $-28y^2$, and whose sum is $-3y$. It is evident that both of these terms contain y as a factor. Therefore we have only to find their numerical coefficients.

Since their product is negative, one must be positive and the other negative; and since their sum is negative, the negative number must have the greater absolute value. The possible pairs of factors of -28 are: 1 and -28; 2 and -14; 4 and -7.

But since $$4 + (-7) = -3,$$

the second terms of the binomial factors are $4y$ and $-7y$.

Consequently, $x^2 - 3xy - 28y^2 = (x + 4y)(x - 7y)$.

EXERCISES IV.

Factor the following expressions:

1. $x^2 - 3x + 2$.	**2.** $x^2 + 3x + 2$.	**3.** $x^2 - x - 2$.
4. $x^2 + x - 2$.	**5.** $x^2 + x - 6$.	**6.** $x^2 - x - 6$.
7. $x^2 + 7x + 6$.	**8.** $x^2 - 5x + 6$.	**9.** $x^2 + 10x - 24$.
10. $x^2 - 2x - 24$.	**11.** $x^2 + 5x - 24$.	**12.** $x^2 - 23x - 24$.
13. $x^2 - 5x - 24$.	**14.** $x^2 + 23x - 24$.	**15.** $x^2 + 2x - 24$.

16. $x^2 - 10x - 24.$ **17.** $x^2 + 3x - 40.$ **18.** $x^2 - 18x - 40.$

19. $x^2 + 6x - 40.$ **20.** $x^2 - 39x - 40.$ **21.** $x^2 - 4x - 60.$

22. $x^2 + 7x - 30.$ **23.** $x^2 + 12x + 32.$ **24.** $x^2 - 3x - 40.$

25. $x^2 - 12x + 35.$ **26.** $x^3 - 17x^2 + 72x.$ **27.** $x^2 + 13x - 30.$

28. $6x - x^2 - x^3.$ **29.** $35 + 2x - x^2.$ **30.** $x^4 + 4x^2 - 21.$

31. $x^4 + 8x^2 + 15.$ **32.** $x^4 - 24x^2 + 63.$ **33.** $3x^6 + 39x^3 + 66.$

34. $x^6 - x^3 - 63.$ **35.** $x^{2n} + 6x^n - 112.$ **36.** $x^{2n} - 16x^n + 55.$

37. $x^2 + (a + b)x + ab.$ **38.** $x^2 - (m + n)x + mn.$

39. $x^2 + (p - q)x - pq.$ **40.** $x^2 + (3r - 2s)x - 6rs.$

41. $ax^2 + 7a^2x + 6a^3.$ **42.** $x^2 + 2xy - 15y^2.$

43. $x^2 - 4ax - 12a^2.$ **44.** $x^2 - 7ax + 12a^2.$

45. $2x^3y^2 - 26x^2y^3 + 84xy^4.$ **46.** $x^2 - 11xm + 30m^2.$

47. $x^2z^2 + 12xz - 13.$ **48.** $a^2b^2 - 7ab + 10.$

49. $m^2n^2 - 20mn + 99.$ **50.** $1 - 25xy + 126x^2y^2.$

51. $1 - 23a^2b + 132a^4b^2.$ **52.** $a^4x^2 - 23a^2x + 120.$

53. $x^4y^4 - 7x^2y^2 - 78.$ **54.** $a^4b^6 + 3a^2b^3 - 108.$

55. $a^4b^8 + 5a^2b^4x^2 - 84x^4.$ **56.** $a^{2n}b^{2n} - 2a^nb^nc^2 - 15c^4.$

13. From Ch. V., Art. 8, we have

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

A trinomial which can be factored by this type-form must satisfy the following conditions:

(i.) *One term of the trinomial is the product of the first terms of its binomial factors.*

(ii.) *A second term of the trinomial is the product of the second terms of its binomial factors.*

(iii.) *The remaining term of the trinomial is the sum of the cross-products.*

Ex. 1. Factor $6x^2 + 19x + 10.$

The first terms of the required binomial factors are factors of $6x^2$, the second terms are factors of 10, and the sum of the cross-products is $19x$.

The factors of $6x^2$ are: x and $6x$, $2x$ and $3x$; and the factors of 10 are: 1 and 10, 2 and 5.

The following arrangements represent possible pairs of factors:

$$\begin{array}{cccc} x+1 & x+10 & x+2 & x+5 \\ \underline{6x+10} & \underline{6x+1} & \underline{6x+5} & \underline{6x+2} \\ 16x & 61x & 17x & 32x \end{array}$$

$$\begin{array}{cccc} 2x+1 & 2x+10 & 2x+2 & 2x+5 \\ \underline{3x+10} & \underline{3x+1} & \underline{3x+5} & \underline{3x+2} \\ 23x & 32x & 16x & 19x \end{array}$$

Since the sum of the cross-products in the last arrangement is equal to the middle term of the given trinomial, we have

$$6x^2 + 19x + 10 = (2x+5)(3x+2).$$

Ex. 2. Factor $5x^2 - 6xy - 8y^2$.

The factors of $5x^2$ are x and $5x$; and the factors of $-8y^2$ are: y and $-8y$, $-y$ and $8y$, $2y$ and $-4y$, $-2y$ and $4y$.

$$\begin{array}{c} x-2y \\ \underline{5x+4y} \\ -6xy \end{array}$$

Since the sum of the cross-products in the arrangement on the left is equal to the middle term of the given trinomial, we have

$$5x^2 - 6xy - 8y^2 = (x-2y)(5x+4y).$$

Ex. 3. Factor $10a^4 + a^2b - 21b^2$.

The factors of $10a^4$ are: a^2 and $10a^2$, $2a^2$ and $5a^2$; and the factors of $-21b^2$ are: b and $-21b$, $-b$ and $21b$, $3b$ and $-7b$, $-3b$ and $7b$.

$$\begin{array}{c} 2a^2+3b \\ \underline{5a^2-7b} \\ a^2b \end{array}$$

Since the sum of the cross-products in the arrangement on the left is equal to the middle term of the given trinomial, we have

$$10a^4 + a^2b - 21b^2 = (2a^2+3b)(5a^2-7b).$$

14. The following directions may be observed in fac trinomials which come under this type-form:

(i.) *When all the terms of the trinomial are positive, onl tive factors of the last term are to be tried.*

(ii.) *When the middle term is negative and the last t positive, the factors of the last term must be both negati*

(iii.) *When the middle term and the last term are both ne one factor of the last term must be positive, the other negati*

(iv.) *Select those pairs of factors of the first and last which, by cross-multiplication, give the middle term of the trin*

EXERCISES V.

Factor the following expressions:

1. $6x^2 + x - 12.$ **2.** $6x^2 - x - 12.$

3. $35x^2 + 32x - 12.$ **4.** $35x^2 + x - 12.$

5. $35x^2 + 16x - 12.$ **6.** $35x^2 - 13x - 12.$

7. $2x^2 + 5x + 2.$ **8.** $10 + 16x + 6x^2.$

9. $6 + 13x - 63x^2.$ **10.** $3x^2 + 13x + 12.$

11. $40 + 2x - 2x^2.$ **12.** $25x^3 + 25x^2 - 6x.$

13. $36x^4 - 18x^2 - 10.$ **14.** $12x - 6x^2 - 90x^3.$

15. $10x^2 + 7x - 33.$ **16.** $8x^4 - 19x^2 - 15.$

17. $40 + 6x - 27x^2.$ **18.** $49x^2 - 35x + 6.$

19. $64x^2 - 92x + 30.$ **20.** $6 - 19x + 15x^2.$

21. $6x^2 - 41x - 56.$ **22.** $30x^2 - 89x + 35.$

23. $18x^2 - 3xy - 45y^2.$ **24.** $3a^2 - 5ab - 2b^2.$

25. $abx^2 - (a^2 + b^2)x + ab.$ **26.** $abx^2 + (a^2 - b^2)x - a$

27. $5a^4x^2 - 4a^2xz - 96z^2.$ **28.** $-10a^4 + 7a^2b^2 + 12$

29. $4x^2 - xy - 3y^2.$ **30.** $10a^2 + 11ab - 6b^2.$

31. $9x^{2n} - 4x^n - 5.$ **32.** $2x^{2r+2} - 3x^{r+1} - 2.$

33. $6x^{2m} + x^m y^n - 15y^{2n}.$ **34.** $10(a+b)^2 + 7c(a+b)\cdot$

35. $7(x - y)^2 - 37z(x - y) + 10z^2.$

36. $6(x^2 + y^2)^2 - 9(x^2 + y^2)z^2 - 15z^4.$

37. $2(a^2 - c^2)^2 - 4b(a^2 - c^2) - 6b^2.$

Binomial Type-Forms.

15. From Ch. V., Art. 6, we have

$$a^2 - b^2 = (a + b)(a - b).$$

That is, *the difference of the squares of two numbers can be written as the product of the sum and the difference of the numbers.*

Ex. 1.
$$\begin{aligned} a^2x^2 - \tfrac{1}{4} b^2 &= (ax)^2 - (\tfrac{1}{2} b)^2 \\ &= (ax + \tfrac{1}{2} b)(ax - \tfrac{1}{2} b). \end{aligned}$$

Ex. 2.
$$\begin{aligned} 32\, m^4n - 2\, n^3 &= 2\, n\,(16\, m^4 - n^2) \\ &= 2\, n\,[(4\, m^2)^2 - n^2] \\ &= 2\, n\,(4\, m^2 + n)(4\, m^2 - n). \end{aligned}$$

16. The difference of any even powers of two numbers can be written as the difference of the squares of two numbers, and should therefore first be factored by applying this type-form.

Ex.
$$\begin{aligned} a^4 - b^4 &= (a^2)^2 - (b^2)^2 \\ &= (a^2 + b^2)(a^2 - b^2) \\ &= (a^2 + b^2)(a + b)(a - b). \end{aligned}$$

EXERCISES VI.

Factor the following expressions:

1. $x^2 - 1$. **2.** $4 - a^2$. **3.** $16 - y^2$.

4. $25\, x^2y^2 - 9$. **5.** $36\, a^2 - 49\, b^2$. **6.** $4\, x^2 - y^4$.

7. $86^2 - 14^2$. **8.** $57^2 - 43^2$. **9.** $37^2 - 27^2$.

10. $81\, a^4 - 16$. **11.** $\frac{4}{9}\, a^2b^2 - \frac{25}{49}\, c^2d^2$. **12.** $16\, a^6 - 25\, b^4c^6$.

13. $a^2b^4c^6 - \frac{1}{4}$. **14.** $\frac{1}{9}\, a^2n^4 - \frac{1}{100}\, x^6$. **15.** $a^{2n} - 1$.

16. $a^{2n} - b^{2m}$. **17.** $x^{2n+2} - 4$. **18.** $9\, a^{2n}b^2 - 4\, c^{2m}$.

19. $7 - 112\, x^4$. **20.** $16\, x^4 - y^4$. **21.** $a^8 - b^8$.

22. $1 - 256\, x^8y^8$. **23.** $x^{16} - y^{16}$. **24.** $a^{16} - 1$.

25. $5\, a^2 - 180\, b^4$. **26.** $\frac{2}{4}\, ab^2 - \frac{2}{9}\, ac^2$. **27.** $\frac{5}{4}\, xy^4 - \frac{5}{25}\, xz^6$.

28. $75\, a^2b^4 - 108\, c^2d^4$. **29.** $243\, b^5c^6 - 75\, b^7$.

30. $a^{4x} - b^{4x}$. **31.** $144\, x^n - x^{n+2}$. **32.** $4\, a^{3n+3} - a^{n+1}$.

33. $a^2 - b^2 + (a + b)c.$ **34.** $a^2 - x^2 + a - x.$

35. $a^4 - a^3 + a - 1.$ **36.** $x^2 - xz - yz - y^2.$

37. $a^2 - a^2n + an^2 - n^2.$ **38.** $a^4 - 2\,ab^3 - b^4 + 2\,a^3b$

39. $x^3y - xy^3 + x^2y + xy^2.$ **40.** $x^2 + 3\,x^3 - x^4 - 3\,x.$

41. $(a + n)(a^2 - x^2) - (a - x)(a^2 - n^2).$

42. $(n - x)(5n^2 - 4x^2) - (3\,x^2 - 4\,n^2)(x - n).$

17. This type-form may frequently be applied to mı nomials.

Ex. **1.** $x^2 - 4\,xy + 4\,y^2 - 9\,z^2 = (x - 2\,y)^2 - (3\,z)^2$
$= (x - 2\,y + 3\,z)(x - 2\,y -$

Ex. **2.** $4\,a^2c^2 - (a^2 - b^2 + c^2)^2$
$= (2\,ac + a^2 - b^2 + c^2)(2\,ac - a^2 + b^2 - c^2)$
$= [(a + c)^2 - b^2][b^2 - (a - c)^2]$
$= (a + c + b)(a + c - b)(b + a - c)(b - a +$

EXERCISES VII.

Factor the following expressions:

1. $(a + b)^2 - c^2.$ **2.** $(a - b)^2 - c^2.$ **3.** $(n + 1)^2 -$

4. $n^2 - (n - 1)^2.$ **5.** $9 - (3 - x)^2.$ **6.** $49 - 4(a +$

7. $(2\,a + b)^2 - 9\,c^2.$ **8.** $(4\,x - 3)^2 - 16\,x^2.$

9. $25\,a^2 - 4(b + c)^2.$ **10.** $36\,x^2 - 81(x - 2)^2.$

11. $(a + b)^2 - (c + d)^2.$ **12.** $(a - b)^2 - (c - d)^2.$

13. $(a + b)^2 - (a - b)^2.$ **14.** $(x + 2)^2 - (x - 1)^2.$

15. $(5\,x - 2)^2 - (4\,x - 3)^2.$ **16.** $(3\,xy - 4)^2 - (2\,xy -$

17. $(a + b - c)^2 - (a - b + c)^2.$ **18.** $(x+y-3)^2-(x-y+$

19. $(x^2 + x + 1)^2 - (x^2 - x + 1)^2.$

20. $(a + b)^2 - 1 - 2(a + b + 1).$

21. $(a - 2\,b)^2 - 9 - 3(a - 2\,b + 3).$

22. $x^2 - 2\,xy + y^2 - z^2.$ **23.** $a^2 - 2\,ab + b^2 - c^2.$

24. $z^2 - x^2 - 2\,xy - y^2.$ **25.** $9 - x^2 + 2\,xy - y^2.$

26. $a^2-n^2+2np-p^2$.

27. $a^2+2bc-b^2-c^2$.

28. $25+12xy-9x^2-4y^2$.

29. $25x^2-49y^2-10x+1$.

30. $a^2-2ab+b^2-x^2-2xy-y^2$.

31. $x^2-2x+1-a^2+2ab-b^2$.

32. $a^2+b^2-c^2-d^2+2ab+2cd$.

33. $x^2+y^2-2xy-a^2-9b^2+6ab$.

34. $25x^2-25b^2+1-a^2-10x+10ab$.

35. $a^4-25a^2-9b^2-30ab-6a^2+9$.

36. $4a^4+9b^4-25c^4+12a^2b^2$.

37. $a^2+b^2-c^2-d^2+2(ab+cd)$.

38. $a^2+b^2-c^2-d^2-2(ab-cd)$.

39. $2(ab+cd)-(a^2+b^2-c^2-d^2)$.

40. $a^2-b^2+2bz-2ax+x^2-z^2$.

41. $4a^2b^2-(a^2+b^2-c^2)^2$.

42. $a^{2r}-a^{4r}-2a^{7r}-a^{10r}$.

43. $a^4+4a^2c-4b^2+4bd^2+4c^2-d^4$.

44. $4(ad+bc)^2-(a^2-b^2-c^2+d^2)^2$.

18. From Ch. V., Art. 10, we derive

$$\boldsymbol{a^3+b^3=(a+b)(a^2-ab+b^2),}$$
$$\boldsymbol{a^3-b^3=(a-b)(a^2+ab+b^2).}$$

Ex. **1.**

$$\begin{aligned} x^3+8y^3&=x^3+(2y)^3\\ &=(x+2y)[x^2-x(2y)+(2y)^2]\\ &=(x+2y)(x^2-2xy+4y^2). \end{aligned}$$

Ex. **2.**

$$\begin{aligned} 512x^6-y^3&=(8x^2)^3-y^3\\ &=(8x^2-y)[(8x^2)^2+8x^2\times y+y^2]\\ &=(8x^2-y)(64x^4+8x^2y+y^2). \end{aligned}$$

Ex. **3.**

$$\begin{aligned} a^6-729b^6&=(a^3)^2-(27b^3)^2\\ &=(a^3+27b^3)(a^3-27b^3)\\ &=(a+3b)(a^2-3ab+9b^2)(a-3b)(a^2+3ab+9b^2). \end{aligned}$$

Ex. 4.

$$
\begin{aligned}
(1-x)^3 - 8x^3 &= (1-x)^3 - (2x)^3 \\
&= (1-x-2x)[(1-x)^2 + (1-x)(2x) + (2x)^2] \\
&= (1-3x)(1+3x^2).
\end{aligned}
$$

19. The sum of the like even powers of two numbers, whose exponents are divisible by an odd number, except 1, can be factored by applying the type-forms of Art. 18.

Ex.
$$
\begin{aligned}
x^{12} + y^{12} &= (x^4)^3 + (y^4)^3 \\
&= (x^4 + y^4)[(x^4)^2 - (x^4)(y^4) + (y^4)^2] \\
&= (x^4 + y^4)(x^8 - x^4y^4 + y^8).
\end{aligned}
$$

EXERCISES VIII.

Factor the following expressions:

1. $x^3 + 1.$	**2.** $x^3 - 8.$	**3.** $a^3 + 27.$
4. $64x^3 - 1.$	**5.** $8x^3 - y^6.$	**6.** $8x^3y^3 - 27.$
7. $125x^3y^6 + 8.$	**8.** $3a^2 - 24a^5.$	**9.** $27a - a^4b^6.$
10. $27x^3 - y^9.$	**11.** $125x^3 - y^{12}z^{12}.$	**12.** $2x^3y^5 + 432y^2.$
13. $27a^3b^3c^6 + 1.$	**14.** $64x^3y^6z^9 - 125.$	**15.** $8m^6n^9 - 343p^9.$
16. $x^6 - 64.$	**17.** $x^6 + y^6.$	**18.** $x^9 + y^9.$
19. $x^9 - 1.$	**20.** $a^{12} - 1.$	**21.** $a^{12} + b^{12}.$
22. $1 - z^{18}.$	**23.** $x^{18} + y^{18}.$	**24.** $a^{3n} - b^{3n}.$

25. $8x^{3n}y^m - 729y^{m+3}z^6.$	**26.** $(x+y)^3 - 1.$
27. $1 - (x-y)^3.$	**28.** $27 - (3+2x)^3.$
29. $(a+b)^3 + (a-b)^3.$	**30.** $(2x-1)^3 - (x-2)^3.$
31. $(2a+x)^3 + (a-2x)^3.$	**32.** $(a+b)^3 - (c+d)^3.$
33. $x^3 - y^3 - 2x^2y + 2xy^2.$	**34.** $4 - x^2 + 4x^3 - x^5.$
35. $x^5 - x^3 - x^2 + 1.$	**36.** $x^3 - 8 - 6x^2 + 12x.$
37. $a^3 - 4a^2c - 4ac^2 + c^3.$	**38.** $n^6 + 5n^4x^2 + 5n^2x^4 + x^6.$

20. From Ch. V., Art. 11, we derive:

(i.) *The sum of the like odd powers of two numbers contains the sum of the numbers as a factor.*

(ii.) *The difference of the like odd powers of two numbers contains the difference of the numbers as a factor.*

Ex. 1. $x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.

Ex. 2. $x^7 - y^7 = (x - y)(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6)$.

EXERCISES IX.

Factor the following expressions:

1. $a^5 + b^5$.	2. $x^5 - 1$.	3. $x^7 + y^7$.
4. $a^7 - 1$.	5. $32\,a^5 - b^{10}$.	6. $243\,x^{10} - y^5$.
7. $a^{10} + b^{10}$.	8. $x^{10} - 1$.	9. $x^{15} + 1$.
10. $128\,x^7 + 1$.	11. $a^5b^5 + 32$.	12. $x^5y^{10} - 1024\,z^{10}$.

Special Devices for Factoring.

21. A factorable expression can frequently be brought to some known type-form by adding to or subtracting from it one or more terms.

Ex. 1. Factor $x^4 + x^2y^2 + y^4$.

This expression would be the square of $x^2 + y^2$, if the coefficient of x^2y^2 were 2. We therefore add x^2y^2; and, in order that the value of the expression may remain the same, we subtract x^2y^2. We then have

$$\begin{aligned} x^4 + 2\,x^2y^2 + y^4 - x^2y^2 &= (x^2 + y^2)^2 - x^2y^2 \\ &= (x^2 + y^2 + xy)(x^2 + y^2 - xy). \end{aligned}$$

22. Another device consists in separating a term into two or more terms, and grouping these component terms with others of the given expression.

Ex. Factor $x^3 - 3\,x^2 + 4$.

Separating $-3\,x^2$ into $-2\,x^2$ and $-x^2$, we obtain

$$\begin{aligned} x^3 - 3\,x^2 + 4 &= x^3 - 2\,x^2 - x^2 + 4 \\ &= x^2(x - 2) - (x^2 - 4) \\ &= (x - 2)[x^2 - (x + 2)] \\ &= (x - 2)(x^2 - x - 2) \\ &= (x - 2)(x - 2)(x + 1) \\ &= (x - 2)^2(x + 1). \end{aligned}$$

EXERCISES X.

Factor the following expressions:

1. $1 + 4x^4$. 2. $1 + 64x^4$. 3. $x^{4n} + 4y^{4n}$.
4. $1 + 3a^2 + 4a^4$. 5. $1 - 7a^2 + a^4$. 6. $1 + 2x^2y^2 + 9x^4$
7. $x^4 - x^2y^2 + 16y^4$. 8. $x^4 + y^4 - 11x^2y^2$. 9. $16x^4 - x^2y^2 + y^4$
10. $x^4 + 4y^4 - 12x^2y^2$. 11. $x^4 + y^8 + x^2y^4$.
12. $x^8 + y^8 - 142x^4y^4$. 13. $x^3 - 6x^2 + 16$.
14. $x^3 - 15x^2 + 250$. 15. $x^3 + 6x^2 + 10x + 4$.
16. $x^3 - 9x^2 + 32x - 42$. 17. $x^3 - 15x^2 + 72x - 110$.
18. $8x^3 - 36x^2 + 48x - 18$.

EXERCISES XI.

Factor the following expressions by the methods given this chapter:

1. $a^4 + 2a^3b - 2ab^3 - b^4$. 2. $ax^2 + (a + b + c)x + b + c$
3. $10c^{4n+1} - 5c^{7n+1} - 5c^{n+1}$. 4. $x^2y^2 + 17xy + 16$.
5. $x^6 + 64$. 6. $a^6b^6 + 1$. 7. $2^{3x+3} - 64$. 8. $x^5y^5 -$
9. $2a^4 - 16ab^3$. 10. $x^4 + 2x^2 + 9$.
11. $24x^2 - (3b - 8a)x - ab$. 12. $b^2 - c^2 + a(a - 2b)$.
13. $x^{2m-2} + 2x^{m+n} + x^{2n+2}$. 14. $x^4 - 2x^3 - 1 + 2x$.
15. $x^2 + 11x + 24$. 16. $a^2 - ab - 6b^2$.
17. $x^2y^2 - 4xy - 5$. 18. $x^2 + x + y - y^2$.
19. $ab(x^2 + y^2) + xy(a^2 + b^2)$.
20. $28(x + 3)^2 - 23(x^2 - 9) - 15(x - 3)^2$.
21. $ax^5 + bx^4 + cx^3 - ax^2 - bx - c$.
22. $(a + b)x^2 + (a - 2b)x - 3b$.
23. $a^2 - b^2 - c^2 - 2a + 2bc + 1$.
24. $49x^4y^6 + 42x^7y^9 + 9x^{10}y^{12}$.
25. $x^2 - 13xy + 40y^2$. 26. $a^2 - 5ab + 6b^2$.
27. $m^2n^2 + 6mn - 55$. 28. $b^2 + ac - c^2 + ab$.

29. $xy - xz + 2yz - y^2 - z^2$.
30. $x^2 - 2x + 1 - y^2$.
31. $15x^2 + x - 40$.
32. $x^3 - x^2z + xz^2 - z^3$.
33. $a^3 - 1 + c - ac$.
34. $a^2 - a - 1 - a^2c + ac + c$.
35. $2a^2 + a - 4ax - x + 2x^2$.
36. $20x^2 - 123x + 180$.
37. $x^3 - 5x^2 - x + 5$.
38. $x^2(x+1) - b^2(b+1)$.
39. $25a^4b^4 + 70a^2b^2c^2 + 49c^4$.
40. $x^4y + zx^3 - xy - z$.
41. $x^2 - 9z^2 - 4y(y + 3z)$.
42. $x^8 - 2x^4y^4 + y^8 - 4x^2y^2(x^2 - y^2)^2$.
43. $a^3 + a^2c + abc + b^2c - b^3$.
44. $5a^4 - 10a^3 - 75a^2$.
45. $3(a-1)^3 - (1-a)$.
46. $x^6 - y^6 + 1 - 2x^3$.
47. $x^2 - ax - bx + ab$.
48. $x^2y^2 + 25 - 9z^2 - 10xy$.
49. $x^2 + 9 - 2x(3 + 2xy^2)$.
50. $a^2b^2 - 4ab - 21$.
51. $3x^6 + 8x^4 - 8x^2 - 3$.
52. $7a^3x^2 + 49a^2x + 84a$.
53. $(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$.
54. $cd - bd + a(b-c)$.
55. $(x^2+1)^3 - (y^2+1)^3$.
56. $abx^3 + x + ab + 1$.
57. $36a^4 - 21a^2 + 1$.
58. $10x^4 - 47x^2 + 42$.
59. $(x^2 + xy - y^2)^2 - (x^2 - xy - y^2)^2$.
60. $x^2 + c(a+b)x + ab(a+c)(c-b)$.
61. $5a^2 - 180b^2$.
62. $\frac{7}{25}abc^2 - \frac{7}{16}abd^2$.
63. $10x^2 + 3x - 18$.
64. $x^{2n} - y^{2n} + 4y^n - 4x^n$.
65. $ab(x^2 - y^2) + xy(a^2 - b^2)$.
66. $36a^4b^2 - 60a^3b^3 + 25a^2b^4$.
67. $a^2(a^2-1) - b^2(b^2-1)$.
68. $(m-n)^2 - 12(m-n) + 27$.
69. $a^2x^5(a^3 - x) - a^5x^2(x^3 - a)$.
70. $(a^2 - b^2)(a+b) + 2ab^2 - 2a^2b$.
71. $(a-b)^2 - x^2 - (x - a + b)(a + b - x)$.
72. $(x+y)^2 - 18(x+y) + 77$.
73. $(a^2 - b^2)x^2 - (a^2 + b^2)x + ab$.
74. $300abc^2 - 432abd^2$.
75. $75a^2b^2 - 108c^2d^2$.
76. $\frac{10}{25}abx^2y^2 - \frac{10}{49}abz^2$.
77. $18a^2x^2 - 98b^2y^2$.
78. $18(x+y)^2 + 23(x^2 - y^2) - 6(x-y)^2$.
79. Express $(a^2 - b^2)(c^2 - d^2)$ as the difference of two squares.

HIGHEST COMMON FACTORS.

23. If two or more integral algebraic expressions have no common factor except 1, they are said to be *prime to one another.*

E.g., ab and cd; $5x^2y$ and $8z^3$; a^2+b^2 and a^2-b^2.

24. The **Highest Common Factor (H. C. F.)** of two or more integral algebraic expressions is the expression of highest degree which exactly divides each of them.

E.g., the H. C. F. of ax^2, bx^3, and cx^4 is evidently x^2.

25. Monomial Expressions. — The H. C. F. of monomials can be found by inspection.

Ex. 1. Find the H. C. F. of x^2y^5z, $x^4y^3z^2$, and $x^3y^4z^4$.

In the expression of highest degree which exactly divides each of the given expressions, the highest power of x is evidently x^2, of y is y^3, and of z is z. Therefore the required H. C. F. is x^2y^3z.

Observe that the power of each letter in the H. C. F. is the *lowest* power to which it occurs in any of the given expressions.

If the monomials contain numerical factors, the Greatest Common Measure (G. C. M.) of these factors should be found as in Arithmetic.

Ex. 2. Find the H. C. F. of $18a^4b^5c^3d$, $42a^3bc^4$, and $30a^2b^2c^2$.

The G. C. M. of the numerical coefficients is 6. The lowest power of a in any of the given expressions is a^2; the lowest power of b is b; the lowest power of c is c^2; and d is not a common factor. Therefore the required H. C. F. is $6a^2bc^2$.

26. In general, to obtain the H. C. F. of two or more monomials:

Multiply the G. C. M. of their numerical coefficients by the product of their common literal factors, each to the lowest power to which it occurs in any of the given monomials.

27. Multinomial Expressions. — The method of finding the H. C. F. of multinomials by factoring is similar to that of finding the H. C. F. of monomials.

Ex. 1. The expressions

$$x^2-1=(x-1)(x+1),$$

and

$$x^2+x-2=(x-1)(x+2),$$

have only the common factor $x-1$. This is their H. C. F.

In general, *the H. C. F. of two or more multinomial expressions is the product of their common factors, each to the lowest power to which it occurs in any of them.*

Ex. 2. Find the H. C. F. of $a^2x^2-a^2$, $2\,ax^2+2\,ax-4\,a$, and $4\,ax^2-12\,ax+8\,a$.

We have

$$a^2x^2-a^2=a^2(x+1)(x-1),$$

$$2\,ax^2+2\,ax-4\,a=2\,a(x+2)(x-1),$$

$$4\,ax^2-12\,ax+8\,a=4\,a(x-2)(x-1).$$

Therefore the required H. C. F. is $a(x-1)$.

EXERCISES XII.

Find the H. C. F. of each of the following expressions:

1. $36\,a^3$, $27\,a^4$.
2. $20\,ab^3$, $35\,a^2b$.
3. $45\,x^2y^3$, $12\,x^3yz$.
4. a^2bx^3, $a^3b^2x^2$, ab^3x^4.
5. $56\,x^4y^3$, $70\,x^2y^5$, $98\,x^3y^2$.
6. $24\,a^2bx^4$, $42\,ax^3$, $18\,a^3x^2y$.
7. $15\,m^4n^3y^2$, $40\,m^2n^4x$, $35\,m^3nx^2$.
8. $9\,(x+y)$, $6\,(x+y)^2$.
9. $12\,y^2(a-b)$, $30\,y\,(a-b)^2$.
10. x^2-9, $x^2+3\,x$.
11. $3\,x^2-3\,xy$, $5\,x-5\,xy^2$.
12. $(a+b)^2$, a^2-b^2.
13. ax^2-a, $ax^2+2\,ax+a$.
14. $x^2-25\,y^2$, $x^2+xy-30\,y^2$.
15. $(a^2b-ab^2)^2$, $ab\,(a^2-b^2)$.
16. $27\,x^3+y^3$, $9\,x^2-y^2$.
17. $a^3-4\,ab^2$, $a^3-8\,b^3$.
18. $x^2-2\,x-15$, $x^2+10\,x+21$.
19. $x^2-2\,x-24$, $x^2+9\,x+20$.
20. $3\,x^3-3\,y^3$, $x^2-by+bx-xy$.
21. x^3-y^3, $x^4+3\,x^2y^2-4\,y^4$.
22. $x^2+xy-30\,y^2$, $x^2-2\,xy-15\,y^2$.
23. $x^2y^2-xy^3-42\,y^4$, $6\,x^3y+18\,x^2y^2-108\,xy^3$.
24. $3\,x^2-ax-4\,a^2$, $6\,x^2-17\,ax+12\,a^2$.
25. $3\,x^3-8\,x^2+4\,x$, $x^3-6\,x^2+12\,x-8$.

26. $a^3 + 2a^2 + 2a + 1,\ a^3 + 1.$

27. $x^2 + ab - ax - bx,\ x^2 - ab - ax + bx.$

28. $a^2 - (b - c)^2,\ (a - c)^2 - b^2.$

29. $x^3 - y^3,\ x^4 + x^2y^2 + y^4.$

30. $x^2 - 3x,\ x^2 - 9,\ x^2 - 6x + 9.$

31. $x^3 - 8,\ x^2 + 7x - 18,\ x^2 - 8x + 12.$

32. $x^2 - 3x - 40,\ x^2 + 3x - 10,\ x^2 - x - 30.$

33. $x^2 + 2xy + y^2 - z^2,\ ax + ay + az.$

34. $(y - z)^2 - x^2,\ (x + y)^2 - z^2,\ y^2 - (z - x)^2.$

LOWEST COMMON MULTIPLES.

28. A **Multiple** of an integral algebraic expression is an expression which is exactly divisible by the given one.

E.g., multiples of $a + b$ are $2(a + b)$, $(x - y)(a + b)$, etc.

29. The **Lowest Common Multiple (L. C. M.)** of two or more integral algebraic expressions is the integral expression of lowest degree which is exactly divisible by each of them.

E.g., the L. C. M. of ax^2, bx^3, and cx^4 is evidently $abcx^4$.

30. Ex. 1. Find the L. C. M. of a^3b, a^2bc^2, and ab^2c^4.

In the expression of lowest degree which is exactly divisible by each of the given expressions, the lowest power of a is evidently a^3, of b is b^2, and of c is c^4. Therefore their L. C. M. is $a^3b^2c^4$.

Observe that the power of each letter in the L. C. M. is the *highest* power to which it occurs in any of the given expressions. If the expressions contain numerical factors, the L. C. M. of these factors should be found as in Arithmetic.

Ex. 2. Find the L. C. M. of

$$3ab^2,\ 6b(x + y)^2, \text{ and } 4a^2b(x - y)(x + y).$$

The L. C. M. of the numerical coefficients is 12.

The highest power of a in any of the expressions is a^2; of b is b^2; of $x + y$ is $(x + y)^2$; and of $x - y$ is $x - y$.

Consequently the required L. C. M. is $12a^2b^2(x + y)^2(x - y)$.

31. In general, to obtain the L. C. M. of two or more monomials:

Multiply the L. C. M. of their numerical coefficients by the product of all the different prime factors of the expressions, each to the highest power to which it occurs in any of them.

EXERCISES XIII.

Find the L. C. M. of the following expressions:

1. $3a,\ 5b$.
2. $3xy^3,\ 8x^2y^2$.
3. $8a^2b,\ 12a^2c^2,\ 10ad$.
4. $30a^3b^4,\ 45a^4b^3,\ 72a^2b^2$.
5. $12x^2y^3,\ 18x^4y^2,\ 36x^5y^4$.
6. $15a^2b^3,\ 60a^3x^2,\ 72b^4x^3$.
7. $40a^3b^4x^5,\ 62a^2b^3x^2,\ 124a^4b^2x^4$.
8. $56m^2nx,\ 72m^4n^2y^4,\ 90m^5x^2y^3$.
9. $3x,\ 5x^2+10x$.
10. $6mn,\ 4m^2-12mn$.
11. $x^2-1,\ x+1$.
12. $3a-6b,\ a^2c-4b^2c$.
13. $x+1,\ x^2-2x-3$.
14. $ax-bx,\ a^3-2a^2b+ab^2$.
15. $(a+b)^2,\ a^2-b^2$.
16. $x^2(m-n),\ x(m^3-n^3)$.
17. $x^2+3x-10,\ x^2-3x-40$.
18. $x^2+6x-55,\ x^2-11x+30$.
19. $x^2-4ax+3a^2,\ x^2+2ax-3a^2$.
20. $m^2+2mn-15n^2,\ m^2+3mn-10n^2$.
21. $a^3-x^3,\ a^2-x^2,\ x-a$.
22. $x^2-y^2,\ (x-y)^2,\ x^3-y^3$.
23. $x-a,\ a^2-x^2,\ x^4-a^4$.
24. $1-2x,\ 4x^2-1,\ 1+4x^2$.
25. $x^2-11x+24,\ x^2-6x-16,\ x^2-x-6$.
26. $x^2-4x-45,\ x^2-7x-18,\ x^2+7x+10$.
27. $3x^2+24x+45,\ 6x^2+18x-60,\ 8x^2-24x+16$.
28. $4x^2+4x-224,\ 6x^2+24x-462,\ 8x^2+64x-264$.
29. $x^2-4ax+3a^2,\ x^2+4ax-5a^2,\ x^2+2ax-15a^2$.
30. $x^2+2mx-3m^2,\ x^2+7mx-8m^2,\ x^2-6mx-27m^2$.
31. $x^2-4a^2,\ x^3+2ax^2+4a^2x+8a^3,\ x^3-2ax^2+4a^2x-8a^3$
32. $a^2-(b+c)^2,\ b^2-(a+c)^2,\ c^2-(a+b)^2$.

H. C. F. AND L. C. M. BY DIVISION.

32. If the given expressions cannot be readily factored, their H. C. F. can be obtained by a method analogous to that used in Arithmetic to find the G. C. M. of numbers.

33. The expressions whose H. C. F. is required should be arranged to powers of a common letter of arrangement.

If one of two expressions be divisible without a remainder by the other, which must be of the same or lower degree in the letter of arrangement, then the latter (the divisor) is the required H. C. F.

For it is a factor of the other expression.

But if the one expression be not divisible without a remainder by the other, their H. C. F. is found as follows:

(i.) *Divide the expression of higher degree in a common letter of arrangement by the one of lower degree; if the expressions be of the same degree, either may be taken as the first divisor.*

(ii.) *Continue the division until the remainder is of lower degree than the divisor in the letter of arrangement.*

(iii.) *Divide the first divisor by the first remainder, the first remainder (second divisor) by the second remainder, and so on, until a remainder 0 is obtained. The last divisor will be the required H. C. F.*

34. Ex. Find the H. C. F. of $2x^3-5x^2-5x+8$ and x^2-4x+3.

We have

$$
\begin{array}{l}
x^2-4x+3)\,2x^3-5x^2-\ \,5x+8\,(2x+3 \\
\qquad\qquad\quad\ \underline{2x^3-8x^2+\ \,6x} \\
\qquad\qquad\qquad\quad\ \ 3x^2-11x \\
\qquad\qquad\qquad\quad\ \ \underline{3x^2-12x+9} \\
\qquad\qquad\qquad\qquad\qquad x-1)\,x^2-4x+3\,(x-3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\ \ \underline{x^2-\ \ x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\ -3x \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \underline{-3x+3}
\end{array}
$$

By Art. 33 (iii.), the H. C. F. is $x-1$.

Principle of H. C. F.

35. The validity of the preceding method is based upon the following principle:

If an integral algebraic expression be divided by another (of the same or lower degree in a common letter of arrangement) and if there be a remainder, then the H. C. F. of this remainder and the divisor is the H. C. F. of the given expressions.

E.g., the H. C. F. of

$$x^4-10x^3+35x^2-50x+24, =(x-1)(x-2)(x-3)(x-4), \quad (1)$$

and $$x^3-7x^2+11x-5, =(x-1)(x-1)(x-5) \quad (2)$$

is evidently $x-1$.

The remainder obtained by dividing (1) by (2) is

$$3x^2-12x+9, =3(x-1)(x-3). \quad (3)$$

The H. C. F. of this remainder and the divisor (2) is evidently also $x-1$, the H. C. F. of (1) and (2).

Notice that the H. C. F. of the remainder and the dividend (1) is $(x-1)(x-3)$, and is *not* the H. C. F. of (1) and (2).

Since this principle can be applied at any stage of the work, the H. C. F. of *any* remainder and the corresponding divisor is the required H. C. F.

When the last remainder is 0, the last divisor is the H. C. F. of itself and the corresponding divisor, that is, of the preceding remainder and divisor, and is, therefore, the required H. C. F.

If a remainder which does not contain the letter of arrangement, and which is not 0, is obtained, the given expressions do not have a H. C. F. in this letter of arrangement.

36. The following principle will frequently simplify the work of finding the H. C. F. of two expressions:

Either of the expressions may be multiplied or divided by any number which is not already a factor of the other expression.

For a factor introduced by multiplication into one expression will not be common to both of them, and therefore will not be introduced into their H. C. F.

In like manner, the factor removed by division from one expression was not common to both of them, and therefore would not have been a factor of their H. C. F.

Ex. Find the H. C. F. of $2x^2 + 5x - 3$ and

$$2x^3 + x^2 - 5x + 2.$$

We have

$$\begin{array}{r|rrrr|l} 2x^2+5x-3 & 2x^3 & +\ x^2 & -\ 5x & +2 & x-2 \\ & 2x^3 & +5x^2 & -\ 3x & & \\ \hline & & -4x^2 & -\ 2x & +2 & \\ & & -4x^2 & -10x & -6 & \\ \hline & & & 8x & -4 & \end{array}$$

The next step would introduce fractional coefficients. To avoid these, we divide $8x - 4$ by 4, since 4 is not a factor of $2x^2 + 5x - 3$, and take $2x - 1$ as the divisor of the second stage:

$$\begin{array}{r|rrr|l} 2x-1 & 2x^2 & +5x & -3 & x+3 \\ & 2x^2 & -\ x & & \\ \hline & & 6x & -3 & \\ & & 6x & -3 & \\ \hline \end{array}$$

The required H. C. F. is $2x - 1$.

37. Before proceeding with the division, remove from the given expressions any monomial factors and set aside their H. C. F. as a factor of the required H. C. F.

Ex. Find the H. C. F. of

$$2x^5y^2 - 12x^4y^2 + 12x^3y^2 - 6x^2y^2 + 4xy^2$$
$$= 2xy^2(x^4 - 6x^3 + 6x^2 - 3x + 2),$$
$$6x^5y - 15x^4y + 21x^3y - 12x^2y$$
$$= 3x^2y(2x^3 - 5x^2 + 7x - 4).$$

We set aside xy, the H. C. F. of $2xy^2$ and $3x^2y$, as a factor of the required H. C. F., and find the H. C. F. of the remaining factors by division.

The first of these expressions cannot be divided by the second without introducing fractional coefficients. To avoid

these we multiply the first by 2, *since 2 is not a factor of the other expression.*

$$
\begin{array}{rrrrrl}
2x^3-5x^2+7x-4)\; 2x^4 & -12x^3 & +12x^2 & -6x & +4 & (x+7 \\
2x^4 & -5x^3 & +7x^2 & -4x & & \\
\times(-2)) & -7x^3 & +5x^2 & -2x & +4 & \\
 & 14x^3 & -10x^2 & +4x & -8 & \\
 & 14x^3 & -35x^2 & +49x & -28 & \\
 & \div 5) & 25x^2 & -45x & +20 & \\
\text{2d divisor,} & & 5x^2 & -9x & +4 &
\end{array}
$$

To avoid fractional coefficients in the next stage of the work, we multiply the last divisor by 5:

$$
\begin{array}{rrrrl}
5x^2-9x+4)\; 10x^3 & -25x^2 & +35x & -20 & (2x-7 \\
10x^3 & -18x^2 & +8x & & \\
\times 5) & -7x^2 & +27x & -20 & \\
 & -35x^2 & +135x & -100 & \\
 & -35x^2 & +63x & -28 & \\
 & \div 72) & 72x & -72 & \\
\end{array}
$$

$$
\begin{array}{rrrrl}
\text{3d divisor,} \quad x-1)\; 5x^2 & -9x & +4 & (5x-4 \\
5x^2 & -5x & & \\
 & -4x & & \\
 & -4x & +4 &
\end{array}
$$

To avoid fractional coefficients, we multiplied the partial remainder of the first division by -2, divided the remainder of the first division by 5, multiplied the partial remainder of the second division by 5, and divided the remainder of the second division by 72.

The required H. C. F. is $xy(x-1)$.

38. *If the divisor and dividend at any stage of the work can be factored readily, it is better to find their H. C. F. by factoring than by continuing the method of division.*

Ex. Find the H. C. F. of

$$x^4-10x^3+35x^2-50x+24, \qquad (1)$$

and

$$x^3-7x^2+11x-5. \qquad (2)$$

We have:

$$\begin{array}{r|l}
x^3-7x^2+11x-5 & x^4-10x^3+35x^2-50x+24 \;(x-3 \\
 & \underline{x^4-7x^3+11x^2-5x} \\
 & -3x^3+24x^2-45x \\
 & \underline{-3x^3+21x^2-33x+15} \\
 & \div 3)\underline{3x^2-12x+9} \\
 & x^2-4x+3
\end{array}$$

The remainder x^2-4x+3, $=(x-1)(x-3)$, is readily factored.

Dividing $x^3-7x^3+11x-5$ by $x-1$, we have

$$x^3-7x^2+11x-5=(x-1)(x^2-6x+5)=(x-1)^2(x-5).$$

The H. C. F. of the first remainder and (2), and therefore the required H. C. F., is $x-1$.

Lowest Common Multiple by Means of H. C. F.

39. If the given expressions cannot be readily factored, their L. C. M. can be obtained by first finding their H. C. F.

Ex. Find the L. C. M. of

$$x^3-2x^2-2x^2y+4xy+x-2y \text{ and } x^3-2x^2y+xy^2-2y^3.$$

The H. C. F. of these expressions is found to be $x-2y$.

Consequently the other factors of the given expressions can be found by dividing each of them by their H. C. F. We have

$$x^3-2x^2-2x^2y+4xy+x-2y=(x-2y)(x^2-2x+1),$$
$$x^3-2x^2y+xy^2-2y^3=(x-2y)(x^2+y^2).$$

From the definition of the H. C. F., as also by inspection, we know that these second factors, x^2-2x+1 and x^2+y^2, have no common factor, and therefore that the L. C. M. of the given expressions must contain both of them as factors.

Consequently the required L. C. M. is

$$(x-2y)(x^2+y^2)(x-1)^2.$$

This example illustrates the following principle:

The L. C. M. of two integral algebraic expressions is the product of their H. C. F. by the remaining factors of the expressions.

Relation between H. C. F. and L. C. M.

40. The following example illustrates an important relation between the H. C. F. and the L. C. M. of two integral algebraic expressions.

Ex. The H. C. F. of

$$x^3 - 1 = (x-1)(x^2+x+1)$$

and

$$x^2 - 1 = (x-1)(x+1)$$

is

$$(x-1).$$

The L. C. M. of the same expressions is

$$(x-1)(x+1)(x^2+x+1).$$

The product of the two given expressions is

$$(x-1)(x-1)(x+1)(x^2+x+1) = (\text{H. C. F.}) \times (\text{L. C. M.}).$$

In general,

The product of two integral algebraic expressions is equal to the product of their H. C. F. and their L. C. M.

It follows from this principle that *the L. C. M. of two integral algebraic expressions can be found by dividing their product by their H. C. F.*

EXERCISES XIV.

Find the H. C. F. and L. C. M. of the following expressions:

1. $x^3 + 4x - 5,\ x^3 - 2x^2 + 6x - 5.$
2. $2x^3 + 3x^2 - x - 12,\ 6x^3 - 17x^2 + 2x + 15.$
3. $x^3 - 3x + 2,\ x^3 + 2x^2 - x - 2.$
4. $2x^3 - 17x^2 + 19x - 4,\ 3x^3 - 20x^2 - 10x + 27.$
5. $x^3 - 5x^2 + 9x - 9,\ x^4 - 4x^2 + 12x - 9.$
6. $x^3 - x^2 - 9x + 9,\ x^4 - 4x^2 + 12x - 9.$
7. $x^3 - 3x^2 + 4,\ x^3 - 2x^2 - 4x + 8.$
8. $x^2 - 3x + 2,\ x^4 - 6x^2 + 8x - 3.$
9. $2x^2 + 3x - 2,\ 4x^3 + 16x^2 - 19x + 5.$
10. $x^3 - 3x^2 + 4,\ 3x^3 - 18x^2 + 36x - 24.$
11. $x^3 - (a+b-c)x^2 + (ab-ac-bc)x + abc,$
 $x^3 - (a-b+c)x^2 + (ac-ab-bc)x + abc.$

12. $x^3 + x^2 - 5x + 3,\ 2x^3 + 7x^2 - 9.$

13. $3x^3 - 8x^2 - 36x + 5,\ 9x^3 - 50x^2 + 27x - 10.$

14. $4x^3y^3 - 3x^2y^2 - 4xy + 3,\ 5x^3y^3 + 8x^2y^2 + xy - 14.$

15. $x^3 - 3xy^2 - 2y^3,\ 2x^3 - 5x^2y - xy^2 + 6y^3.$

16. $a^3 - a^2 - 5a + 2,\ 3a^3 - a^2 - 8a + 12.$

17. $x^3 + 2x^2 + 2x + 1,\ x^3 - 4x^2 - 4x - 5.$

18. $30x^3 - 25ax^2 + 8a^2x - a^3,\ 18x^3 - 24ax^2 + 15a^2x - 3a^3.$

19. $2x^4 - 3x^3 + 4x^2 - 5x - 4,\ 2x^4 - x^3 + x - 12.$

20. $4x^3 - 8x^2 + 5x - 3,\ 2x^4 - 3x^3 + 6x^2 - 3x + 2.$

21. $4x^4 - 8x^3 - 3x^2 + 7x - 2,\ 3x^3 - 11x^2 + 2x + 16.$

22. $36a^6 + 9a^3 - 27a^4 - 18a^5,\ 27a^5b^2 - 9a^3b^2 - 18a^4b^2.$

23. $3x^5 - 10x^3 + 15x + 8,\ x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6.$

24. $2x^3 - 3x^2 - 8x - 3,\ 2x^4 - 9x^3 + 13x^2 - 23x - 16.$

25. $x^5 + x^3 - 8x^2 - 8,\ x^4 - 2x^3 + x^2 - 2x.$

The H. C. F. and L. C. M. of Three or More Expressions.

41. To find the H. C. F. of three or more integral algebraic expressions find the H. C. F. of any two of them, next the H. C. F. of that H. C. F. and the third expression, and so on.

42. To find the L. C. M. of three or more integral algebraic expressions, find the L. C. M. of any two of them; next, the L. C. M. of a third and the L. C. M. already found, and so on.

EXERCISES XV.

Find the H. C. F. and the L. C. M. of the following expressions:

1. $x^3 - 4x + 3,\ 2x^3 + x^2 - 7x + 4,\ x^3 - 2x^2 + 1.$

2. $x^3-6x^2+11x-6,\ x^3-9x^2+26x-24,\ x^3-8x^2+19x-12.$

3. $2x^3+5x^2-4x-10,\ 2x^3+5x^2+2x+5,\ 2x^3+7x^2+7x+5.$

4. $2x^4 + 6x^3 + 4x^2,\ 3x^3 + 9x^2 + 9x + 6,\ 3x^3 + 8x^2 + 5x + 2.$

5. $2x^4 - x^3 + 3x^2 + x + 4,\ 2x^4 - 3x^3 - 2x^2 + 9x - 12,$
$4x^4 - 16x^3 + 25x^2 - 23x + 4.$

SOLUTION OF EQUATIONS BY FACTORING.

43. The roots of the equation

$$(x-1)(x-2)=0 \qquad (1)$$

are evidently 1 and 2. For 1 reduces the first member to $0\times(-1), =0$; and 2 reduces the first member to $1\times 0, =0$. Therefore equation (1) is equivalent to the equations

$$x-1=0 \text{ and } x-2=0, \text{ jointly.}$$

This example illustrates the following method of solving an equation by factoring:

Transfer all terms to the first member. Factor this first member, and equate each of the resulting factors to zero. Solve the equations thus obtained.

Ex. 1. Solve the equation $x(x-2)(x+5)=0$.

Equating factors to 0, $x=0$; $x-2=0$, whence $x=2$;

and $x+5=0$, whence $x=-5$.

The roots are therefore 0, 2, and -5.

Ex. 2. Solve the equation $x^2-1=3$.

Transferring 3 to first member, and factoring, we have

$$(x-2)(x+2)=0.$$

Equating factors to 0, $x-2=0$, whence $x=2$;

and $x+2=0$, whence $x=-2$.

The roots are therefore $+2$ and -2.

The statement $+2$ *and* -2 is usually written ± 2, read *positive and negative two.*

Ex. 3. Solve the equation $x^2+2x-12=3$.

Transferring 3, $x^2+2x-15=0$.

Factoring, $(x+5)(x-3)=0$.

Equating factors to 0, $x+5=0$, whence $x=-5$;

and $x-3=0$, whence $x=3$.

The required roots are therefore -5, 3.

EXERCISES XVI.

Solve each of the following equations:

1. $x(x-3)=0.$
2. $x(x+5)=0.$
3. $5x(x+7)=0.$
4. $(x-2)(x+1)=0.$
5. $(3x+2)(5x-3)=0.$
6. $x(x+2)(3x-1)=0.$
7. $3x(16x^2-25)=0.$
8. $(x^2-1)(9x^2-16)=0.$
9. $(x^2-9)(4x^2-25)=0.$
10. $(25x^2-4)(x^2-196)=0.$
11. $x^2-11=5.$
12. $4x^2-15=1.$
13. $23-9x^2=-2.$
14. $5x^2-16=4.$
15. $7x^2-46=5x^2+4.$
16. $x^2-x-2=0.$
17. $x^2+x=12.$
18. $x^2+3x-28=0.$
19. $x^2-x=30.$
20. $2x^2-x-3=0.$
21. $3x^2-13x-10=0.$
22. $10x^2+21x-10=0.$
23. $15x^2+14x-8=0.$
24. $15x^2-22x+8=0.$
25. $(x-5)(x-6)=30.$
26. $(x-12)(x+15)=-180.$
27. $(x+15)(x+4)=60.$
28. $(x+20)(x-5)=-100.$

29. If 24 is added to the square of a number, the sum will be equal to eleven times the number. What is the number?

30. If 40 is added to the square of a number, the sum will be equal to thirteen times the number. What is the number?

31. In a number of 2 digits, the units' digit is 2 greater than the tens' digit. The product of the digits is equal to the number diminished by 16. What is the number?

32. The length of a field exceeds its breadth by 3 rods. If 18 rods were added to its length, and 2 rods were taken from its breadth, the area would be doubled. What are the dimensions of the field?

33. The number of square feet in the area of a square floor, increased by 20, is equal to nine times the number of feet in its side. What is the length of a side of the room?

CHAPTER VII.

FRACTIONS.

1. The quotient of a division can be expressed as an integer or an integral expression only when the dividend is a multiple of the divisor; as $a^2b \div ab = a$; $(ax^2 + 2bx) \div x = ax + 2b$.

If the dividend be not a multiple of the divisor, the quotient is called a **Fraction**; as $a \div b$; $(ax^2 + 2bx) \div x^3$.

2. The notation for a fraction in Algebra is the same as in ordinary Arithmetic.

Thus, $(ax^2 + 2bx) \div x^3$ is written $\dfrac{ax^2 + 2bx}{x^3}$.

The **Solidus**, /, is frequently used instead of the horizontal line to denote a fraction; as $(ax^2 + bx)/x^3$ for $\dfrac{ax^2 + bx}{x^3}$.

3. As in Arithmetic, the dividend is called the **Numerator** of the fraction, the divisor the **Denominator**, and the two are called the **Terms** of the fraction.

4. An integer or an integral expression can be written in a *fractional form* with a denominator 1.

E.g., $$7 = \frac{7}{1}, \quad a + b = \frac{a+b}{1}.$$

It is important to notice that an algebraic fraction may be *arithmetically* integral for certain values of its terms.

E.g., when $a = 4$ and $b = 2$, the fraction a/b becomes $4/2 = 2$.

5. By the definition of a fraction, a/b is a number which, multiplied by b, becomes a; that is,

$$(a/b) \times b = a, \text{ or } \frac{a}{b} \times b = a \qquad (1)$$

6. The Sign of a Fraction. — The sign of a fraction is written before the line separating its numerator from its denominator; as $+\frac{a}{b}$, $-\frac{a}{b}$.

Since a fraction is a quotient, the sign of a fraction is determined by the rule of signs in division.

$$\frac{+a}{+b}=+\frac{a}{b},\quad \frac{-a}{-b}=+\frac{a}{b},\quad \frac{+a}{-b}=-\frac{a}{b},\quad \frac{-a}{+b}=-\frac{a}{b}.$$

7. From the rule of signs we derive:

(i.) *If the signs of the numerator and the denominator of a fraction be reversed, the sign of the fraction is unchanged.*

E.g., $$\frac{-7}{3}=\frac{7}{-3};\quad \frac{x}{x-1}=\frac{-x}{1-x}.$$

This step is equivalent to multiplying or dividing both terms of the fraction by -1.

(ii.) *If the sign of either the numerator or the denominator of a fraction be reversed, the sign of the fraction is reversed; and conversely.*

E.g., $$\frac{7}{3}=-\frac{-7}{3};\quad \frac{-x}{x-1}=-\frac{x}{x-1};\quad -\frac{x-a}{b-x}=\frac{x-a}{x-b}.$$

(iii.) *If the signs of an even number of factors in the numerator and denominator, either or both, of a fraction be reversed, the sign of the fraction is unchanged; but, if the signs of an odd number of factors be reversed, the sign of the fraction is reversed.*

E.g., $$\frac{x-a}{(a-b)(b-c)(c-a)}=-\frac{x-a}{(a-b)(b-c)(a-c)}$$
$$=\frac{x-a}{(b-a)(b-c)(a-c)}$$
$$=\frac{a-x}{(a-b)(b-c)(a-c)}.$$

Reduction of Fractions to Lowest Terms.

8. A fraction is said to be *in its lowest terms* when its numerator and denominator have no common integral factor.

E.g., $$\frac{2}{3}, \frac{x-1}{x^2+1}.$$

9. *The value of a fraction is not changed if both numerator and denominator be divided by the same number, not* 0.

E.g., $$\frac{a+ab}{a+ac}=\frac{(a+ab)\div a}{(a+ac)\div a}=\frac{1+b}{1+c}.$$

Let the value of $\frac{a}{b}$ be denoted by v; or $v=\frac{a}{b}$.

Multiplying by b, $$vb=\frac{a}{b}\times b=a.$$

Dividing by n, $vb\div n=a\div n$, or $v(b\div n)=a\div n$.

Dividing by $b\div n$, $$v=a\div n\div(b\div n),$$
$$=\frac{a\div n}{b\div n}.$$

But $$v=\frac{a}{b}.$$

Therefore $$\frac{a}{b}=\frac{a\div n}{b\div n}.$$

10. Ex. 1. Reduce $\frac{6a^3b^2}{8a^2b^5}$ to its lowest terms.

The factor $2a^2b^2$ is the H. C. F. of the numerator and denominator. We therefore have

$$\frac{6a^3b^2}{8a^2b^5}=\frac{6a^3b^2\div 2a^2b^2}{8a^2b^5\div 2a^2b^2}=\frac{3a}{4b^3}.$$

A fraction is reduced to its lowest terms by dividing its numerator and denominator by the H. C. F. of its terms.

This step is called *cancelling common factors, and can usually be done mentally, if the terms of the fraction are first resolved into their prime factors.*

In like manner, the factor removed by division from one expression was not common to both of them, and therefore would not have been a factor of their H. C. F.

Ex. Find the H. C. F. of $2x^2 + 5x - 3$ and

$$2x^3 + x^2 - 5x + 2.$$

We have

$$\begin{array}{r|l} 2x^2+5x-3) & 2x^3 + x^2 - 5x + 2(x-2 \\ & \underline{2x^3 + 5x^2 - 3x} \\ & - 4x^2 - 2x \\ & \underline{- 4x^2 - 10x + 6} \\ & 8x - 4 \end{array}$$

The next step would introduce fractional coëfficients. To avoid these, we divide $8x - 4$ by 4, since 4 is not a factor of $2x^2 + 5x - 3$, and take $2x - 1$ as the divisor of the second stage:

$$\begin{array}{r|l} 2x-1) & 2x^2 + 5x - 3(x+3 \\ & \underline{2x^2 - x} \\ & 6x - 3 \\ & \underline{6x - 3} \end{array}$$

The required H. C. F. is $2x - 1$.

37. Before proceeding with the division, remove from the given expressions any monomial factors and set aside their H. C. F. as a factor of the required H. C. F.

Ex. Find the H. C. F. of

$$2x^5y^2 - 12x^4y^2 + 12x^3y^2 - 6x^2y^2 + 4xy^2$$
$$= 2xy^2(x^4 - 6x^3 + 6x^2 - 3x + 2),$$
$$6x^5y - 15x^4y + 21x^3y - 12x^2y$$
$$= 3x^2y(2x^3 - 5x^2 + 7x - 4).$$

We set aside xy, the H. C. F. of $2xy^2$ and $3x^2y$, as a factor of the required H. C. F., and find the H. C. F. of the remaining factors by division.

The first of these expressions cannot be divided by the second without introducing fractional coefficients. To avoid

these we multiply the first by 2, *since 2 is not a factor of the other expression.*

$$
\begin{array}{rl}
2x^3-5x^2+7x-4) & 2x^4-12x^3+12x^2-\ \ 6x+4\,(x+7 \\
& \underline{2x^4-\ \ 5x^3+\ \ 7x^2-\ \ 4x} \\
\times(-2)) & -\ \ 7x^3+\ \ 5x^2-\ \ 2x+\ \ 4 \\
& \quad 14x^3-10x^2+\ \ 4x-\ \ 8 \\
& \quad \underline{14x^3-35x^2+49x-28} \\
\div 5) & \quad 25x^2-45x+20 \\
\text{2d divisor,} & \quad 5x^2-\ \ 9x+\ \ 4
\end{array}
$$

To avoid fractional coefficients in the next stage of the work, we multiply the last divisor by 5:

$$
\begin{array}{rl}
5x^2-9x+4) & 10x^3-25x^2+\ \ 35x-\ \ 20\,(2x-7 \\
& \underline{10x^3-18x^2+\ \ 8x} \\
\times 5) & -\ \ 7x^2+\ \ 27x-\ \ 20 \\
& -35x^2+135x-100 \\
& \underline{-35x^2+\ \ 63x-\ \ 28} \\
\div 72) & 72x-\ \ 72 \\
\text{3d divisor,} \quad x-1) & 5x^2-9x+4\,(5x-4 \\
& \underline{5x^2-5x} \\
& -4x \\
& \underline{-4x+4}
\end{array}
$$

To avoid fractional coefficients, we multiplied the partial remainder of the first division by -2, divided the remainder of the first division by 5, multiplied the partial remainder of the second division by 5, and divided the remainder of the second division by 72.

The required H. C. F. is $xy(x-1)$.

38. *If the divisor and dividend at any stage of the work can be factored readily, it is better to find their H. C. F. by factoring than by continuing the method of division.*

Ex. Find the H. C. F. of

$$x^4-10x^3+35x^2-50x+24, \qquad (1)$$

and

$$x^3-7x^2+11x-5. \qquad (2)$$

We have:

$$\begin{array}{r}
x^3-7x^2+11x-5)\,x^4-10x^3+35x^2-50x+24\,(x-3 \\
\underline{x^4-7x^3+11x^2-5x} \\
-3x^3+24x^2-45x \\
\underline{-3x^3+21x^2-33x+15} \\
\div 3)\,\underline{3x^2-12x+9} \\
x^2-4x+3
\end{array}$$

The remainder $x^2-4x+3,=(x-1)(x-3)$, is readily factored.

Dividing $x^3-7x^3+11x-5$ by $x-1$, we have

$$x^3-7x^2+11x-5=(x-1)(x^2-6x+5)=(x-1)^2(x-5).$$

The H. C. F. of the first remainder and (2), and therefore the required H. C. F., is $x-1$.

Lowest Common Multiple by Means of H. C. F.

39. If the given expressions cannot be readily factored, their L. C. M. can be obtained by first finding their H. C. F.

Ex. Find the L. C. M. of

$x^3-2x^2-2x^2y+4xy+x-2y$ and $x^3-2x^2y+xy^2-2y^3$.

The H. C. F. of these expressions is found to be $x-2y$.

Consequently the other factors of the given expressions can be found by dividing each of them by their H. C. F. We have

$$x^3-2x^2-2x^2y+4xy+x-2y=(x-2y)(x^2-2x+1),$$
$$x^3-2x^2y+xy^2-2y^3=(x-2y)(x^2+y^2).$$

From the definition of the H. C. F., as also by inspection, we know that these second factors, x^2-2x+1 and x^2+y^2, have no common factor, and therefore that the L. C. M. of the given expressions must contain both of them as factors.

Consequently the required L. C. M. is

$$(x-2y)(x^2+y^2)(x-1)^2.$$

This example illustrates the following principle:

The L. C. M. of two integral algebraic expressions is the product of their H. C. F. by the remaining factors of the expressions.

Relation between H. C. F. and L. C. M.

40. The following example illustrates an important relation between the H. C. F. and the L. C. M. of two integral algebraic expressions.

Ex. The H. C. F. of

$$x^3 - 1 = (x-1)(x^2+x+1)$$

and

$$x^2 - 1 = (x-1)(x+1)$$

is

$$(x-1).$$

The L. C. M. of the same expressions is

$$(x-1)(x+1)(x^2+x+1).$$

The product of the two given expressions is

$$(x-1)(x-1)(x+1)(x^2+x+1) = (\text{H. C. F.}) \times (\text{L. C. M.}).$$

In general,

The product of two integral algebraic expressions is equal to the product of their H. C. F. and their L. C. M.

It follows from this principle that *the L. C. M. of two integral algebraic expressions can be found by dividing their product by their H. C. F.*

EXERCISES XIV.

Find the H. C. F. and L. C. M. of the following expressions:

1. $x^3 + 4x - 5,\ x^3 - 2x^2 + 6x - 5.$
2. $2x^3 + 3x^2 - x - 12,\ 6x^3 - 17x^2 + 2x + 15.$
3. $x^3 - 3x + 2,\ x^3 + 2x^2 - x - 2.$
4. $2x^3 - 17x^2 + 19x - 4,\ 3x^3 - 20x^2 - 10x + 27.$
5. $x^3 - 5x^2 + 9x - 9,\ x^4 - 4x^2 + 12x - 9.$
6. $x^3 - x^2 - 9x + 9,\ x^4 - 4x^2 + 12x - 9.$
7. $x^3 - 3x^2 + 4,\ x^3 - 2x^2 - 4x + 8.$
8. $x^2 - 3x + 2,\ x^4 - 6x^2 + 8x - 3.$
9. $2x^2 + 3x - 2,\ 4x^3 + 16x^2 - 19x + 5.$
10. $x^3 - 3x^2 + 4,\ 3x^3 - 18x^2 + 36x - 24.$
11. $x^3 - (a + b - c)x^2 + (ab - ac - bc)x + abc,$
    $x^3 - (a - b + c)x^2 + (ac - ab - bc)x + abc.$

12. $x^3 + x^2 - 5x + 3,\ 2x^3 + 7x^2 - 9.$

13. $3x^3 - 8x^2 - 36x + 5,\ 9x^3 - 50x^2 + 27x - 10.$

14. $4x^3y^3 - 3x^2y^2 - 4xy + 3,\ 5x^3y^3 + 8x^2y^2 + xy - 14.$

15. $x^3 - 3xy^2 - 2y^3,\ 2x^3 - 5x^2y - xy^2 + 6y^3.$

16. $a^3 - a^2 - 5a + 2,\ 3a^3 - a^2 - 8a + 12.$

17. $x^3 + 2x^2 + 2x + 1,\ x^3 - 4x^2 - 4x - 5.$

18. $30x^3 - 25ax^2 + 8a^2x - a^3,\ 18x^3 - 24ax^2 + 15a^2x - 3a^3.$

19. $2x^4 - 3x^3 + 4x^2 - 5x - 4,\ 2x^4 - x^3 + x - 12.$

20. $4x^3 - 8x^2 + 5x - 3,\ 2x^4 - 3x^3 + 6x^2 - 3x + 2.$

21. $4x^4 - 8x^3 - 3x^2 + 7x - 2,\ 3x^3 - 11x^2 + 2x + 16.$

22. $36a^6 + 9a^3 - 27a^4 - 18a^5,\ 27a^5b^2 - 9a^3b^2 - 18a^4b^2.$

23. $3x^5 - 10x^3 + 15x + 8,\ x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6.$

24. $2x^3 - 3x^2 - 8x - 3,\ 2x^4 - 9x^3 + 13x^2 - 23x - 16.$

25. $x^5 + x^3 - 8x^2 - 8,\ x^4 - 2x^3 + x^2 - 2x.$

The H. C. F. and L. C. M. of Three or More Expressions.

41. To find the H. C. F. of three or more integral algebraic expressions find the H. C. F. of any two of them, next the H. C. F. of that H. C. F. and the third expression, and so on.

42. To find the L. C. M. of three or more integral algebraic expressions, find the L. C. M. of any two of them; next, the L. C. M. of a third and the L. C. M. already found, and so on.

EXERCISES XV.

Find the H. C. F. and the L. C. M. of the following expressions:

1. $x^3 - 4x + 3,\ 2x^3 + x^2 - 7x + 4,\ x^3 - 2x^2 + 1.$

2. $x^3-6x^2+11x-6,\ x^3-9x^2+26x-24,\ x^3-8x^2+19x-12.$

3. $2x^3+5x^2-4x-10,\ 2x^3+5x^2+2x+5,\ 2x^3+7x^2+7x+5.$

4. $2x^4 + 6x^3 + 4x^2,\ 3x^3 + 9x^2 + 9x + 6,\ 3x^3 + 8x^2 + 5x + 2.$

5. $2x^4 - x^3 + 3x^2 + x + 4,\ 2x^4 - 3x^3 - 2x^2 + 9x - 12,$
$4x^4 - 16x^3 + 25x^2 - 23x + 4.$

SOLUTION OF EQUATIONS BY FACTORING.

43. The roots of the equation

$$(x-1)(x-2)=0 \tag{1}$$

are evidently 1 and 2. For 1 reduces the first member to $0\times(-1)$, $=0$; and 2 reduces the first member to 1×0, $=0$. Therefore equation (1) is equivalent to the equations

$$x-1=0 \text{ and } x-2=0, \text{ jointly.}$$

This example illustrates the following method of solving an equation by factoring:

Transfer all terms to the first member. Factor this first member, and equate each of the resulting factors to zero. Solve the equations thus obtained.

Ex. 1. Solve the equation $x(x-2)(x+5)=0$.

Equating factors to 0, $x=0$; $x-2=0$, whence $x=2$;

and $x+5=0$, whence $x=-5$.

The roots are therefore 0, 2, and -5.

Ex. 2. Solve the equation $x^2-1=3$.

Transferring 3 to first member, and factoring, we have

$$(x-2)(x+2)=0.$$

Equating factors to 0, $x-2=0$, whence $x=2$;

and $x+2=0$, whence $x=-2$.

The roots are therefore $+2$ and -2.

The statement $+2$ *and* -2 is usually written ± 2, read *positive and negative two.*

Ex. 3. Solve the equation $x^2+2x-12=3$.

Transferring 3, $x^2+2x-15=0$.

Factoring, $(x+5)(x-3)=0$.

Equating factors to 0, $x+5=0$, whence $x=-5$;

and $x-3=0$, whence $x=3$.

The required roots are therefore -5, 3.

EXERCISES XVI.

Solve each of the following equations:

1. $x(x-3)=0$.
2. $x(x+5)=0$.
3. $5x(x+7)=0$.
4. $(x-2)(x+1)=0$.
5. $(3x+2)(5x-3)=0$.
6. $x(x+2)(3x-1)=0$.
7. $3x(16x^2-25)=0$.
8. $(x^2-1)(9x^2-16)=0$.
9. $(x^2-9)(4x^2-25)=0$.
10. $(25x^2-4)(x^2-196)=0$.
11. $x^2-11=5$.
12. $4x^2-15=1$.
13. $23-9x^2=-2$.
14. $5x^2-16=4$.
15. $7x^2-46=5x^2+4$.
16. $x^2-x-2=0$.
17. $x^2+x=12$.
18. $x^2+3x-28=0$.
19. $x^2-x=30$.
20. $2x^2-x-3=0$.
21. $3x^2-13x-10=0$.
22. $10x^2+21x-10=0$.
23. $15x^2+14x-8=0$.
24. $15x^2-22x+8=0$.
25. $(x-5)(x-6)=30$.
26. $(x-12)(x+15)=-180$.
27. $(x+15)(x+4)=60$.
28. $(x+20)(x-5)=-100$.

29. If 24 is added to the square of a number, the sum will be equal to eleven times the number. What is the number?

30. If 40 is added to the square of a number, the sum will be equal to thirteen times the number. What is the number?

31. In a number of 2 digits, the units' digit is 2 greater than the tens' digit. The product of the digits is equal to the number diminished by 16. What is the number?

32. The length of a field exceeds its breadth by 3 rods. If 18 rods were added to its length, and 2 rods were taken from its breadth, the area would be doubled. What are the dimensions of the field?

33. The number of square feet in the area of a square floor, increased by 20, is equal to nine times the number of feet in its side. What is the length of a side of the room?

CHAPTER VII.

FRACTIONS.

1. The quotient of a division can be expressed as an integer or an integral expression only when the dividend is a multiple of the divisor; as $a^2b \div ab = a$; $(ax^2 + 2\,bx) \div x = ax + 2\,b$.

If the dividend be not a multiple of the divisor, the quotient is called a **Fraction**; as $a \div b$; $(ax^2 + 2\,bx) \div x^3$.

2. The notation for a fraction in Algebra is the same as in ordinary Arithmetic.

Thus, $(ax^2 + 2\,bx) \div x^3$ is written $\dfrac{ax^2 + 2\,bx}{x^3}$.

The **Solidus**, /, is frequently used instead of the horizontal line to denote a fraction; as $(ax^2 + bx)/x^3$ for $\dfrac{ax^2 + bx}{x^3}$.

3. As in Arithmetic, the dividend is called the **Numerator** of the fraction, the divisor the **Denominator**, and the two are called the **Terms** of the fraction.

4. An integer or an integral expression can be written in a *fractional form* with a denominator 1.

E.g., $$7 = \frac{7}{1},\quad a + b = \frac{a+b}{1}.$$

It is important to notice that an algebraic fraction may be *arithmetically* integral for certain values of its terms.

E.g., when $a = 4$ and $b = 2$, the fraction a/b becomes $4/2 = 2$.

5. By the definition of a fraction, a/b is a number which, multiplied by b, becomes a; that is,

$$(a/b) \times b = a, \text{ or } \frac{a}{b} \times b = a \qquad (1)$$

6. The Sign of a Fraction. — The sign of a fraction is written before the line separating its numerator from its denominator; as $+\frac{a}{b}$, $-\frac{a}{b}$.

Since a fraction is a quotient, the sign of a fraction is determined by the rule of signs in division.

$$\frac{+a}{+b}=+\frac{a}{b},\quad \frac{-a}{-b}=+\frac{a}{b},\quad \frac{+a}{-b}=-\frac{a}{b},\quad \frac{-a}{+b}=-\frac{a}{b}.$$

7. From the rule of signs we derive:

(i.) *If the signs of the numerator and the denominator of a fraction be reversed, the sign of the fraction is unchanged.*

E.g., $$\frac{-7}{3}=\frac{7}{-3};\quad \frac{x}{x-1}=\frac{-x}{1-x}.$$

This step is equivalent to multiplying or dividing both terms of the fraction by -1.

(ii.) *If the sign of either the numerator or the denominator of a fraction be reversed, the sign of the fraction is reversed; and conversely.*

E.g., $$\frac{7}{3}=-\frac{-7}{3};\quad \frac{-x}{x-1}=-\frac{x}{x-1};\quad -\frac{x-a}{b-x}=\frac{x-a}{x-b}.$$

(iii.) *If the signs of an even number of factors in the numerator and denominator, either or both, of a fraction be reversed, the sign of the fraction is unchanged; but, if the signs of an odd number of factors be reversed, the sign of the fraction is reversed.*

E.g., $$\frac{x-a}{(a-b)(b-c)(c-a)}=-\frac{x-a}{(a-b)(b-c)(a-c)}$$
$$=\frac{x-a}{(b-a)(b-c)(a-c)}$$
$$=\frac{a-x}{(a-b)(b-c)(a-c)}.$$

Reduction of Fractions to Lowest Terms.

8. A fraction is said to be *in its lowest terms* when its numerator and denominator have no common integral factor.

E.g., $$\frac{2}{3},\ \frac{x-1}{x^2+1}.$$

9. *The value of a fraction is not changed if both numerator and denominator be divided by the same number, not* 0.

E.g., $$\frac{a+ab}{a+ac}=\frac{(a+ab)\div a}{(a+ac)\div a}=\frac{1+b}{1+c}.$$

Let the value of $\frac{a}{b}$ be denoted by v; or $v=\frac{a}{b}$.

Multiplying by b, $$vb=\frac{a}{b}\times b=a.$$

Dividing by n, $vb\div n=a\div n$, or $v(b\div n)=a\div n$.

Dividing by $b\div n$, $$v=a\div n\div(b\div n),$$ $$=\frac{a\div n}{b\div n}.$$

But $$v=\frac{a}{b}.$$

Therefore $$\boldsymbol{\frac{a}{b}=\frac{a\div n}{b\div n}}.$$

10. Ex. 1. Reduce $\frac{6a^3b^2}{8a^2b^5}$ to its lowest terms.

The factor $2a^2b^2$ is the H. C. F. of the numerator and denominator. We therefore have

$$\frac{6a^3b^2}{8a^2b^5}=\frac{6a^3b^2\div 2a^2b^2}{8a^2b^5\div 2a^2b^2}=\frac{3a}{4b^3}.$$

A fraction is reduced to its lowest terms by dividing its numerator and denominator by the H. C. F. of its terms.

This step is called *cancelling common factors, and can usually be done mentally, if the terms of the fraction are first resolved into their prime factors.*

Ex. 2. $$\frac{a^2-x^2}{(a+x)^2}=\frac{(a+x)(a-x)}{(a+x)(a+x)}=\frac{a-x}{a+x}.$$

Ex. 3. $$\frac{x^2-xy-2y^2}{4y^2-x^2}=\frac{(x-2y)(x+y)}{(2y-x)(2y+x)}.$$

Changing the sign of the first factor in the numerator and the sign of the fraction, we have

$$-\frac{(2y-x)(x+y)}{(2y-x)(2y+x)}=-\frac{x+y}{2y+x}.$$

Ex. 4. Reduce $\frac{x^3-3x^2+3x-2}{x^3-x^2-x-2}$ to its lowest terms.

We find $x-2$ to be the H. C. F. of numerator and denominator by Ch. VI., Art. 33.

Then $$\frac{(x^3-3x^2+3x-2)\div(x-2)}{(x^3-x^2-x-2)\div(x-2)}=\frac{x^2-x+1}{x^2+x+1}.$$

EXERCISES I.

Reduce each of the following fractions to its lowest terms:

1. $\frac{ab}{ac}$. 2. $\frac{a^2x}{ax^2}$. 3. $\frac{a^2x^3}{5a^3x^2}$. 4. $\frac{4x^4m^2n^3}{8x^3m^2n^6}$.

5. $\frac{2a^2b^3c^4}{5a^3b^2c^5}$. 6. $\frac{150a^3x^4z^7}{48a^4x^7}$. 7. $\frac{a^{n+1}b}{a^{n-1}b^m}$. 8. $\frac{5(x+y)^3}{15(x+y)^2}$.

9. $\frac{m-n}{2m-2n}$. 10. $\frac{a^2+ab}{a^2-ab}$. 11. $\frac{15x-9}{6-10x}$. 12. $\frac{x^3-x^2y}{xy^2-y^3}$.

13. $\frac{(x+1)^2}{x^2+x}$. 14. $\frac{2-x}{x^2-4}$. 15. $\frac{5a^2+5ax}{a^2-x^2}$.

16. $\frac{a-b}{a^3-b^3}$. 17. $\frac{ax+bx}{na^2-nb^2}$. 18. $\frac{3x^2-12a^2}{3x+6a}$.

19. $\frac{3b-2a}{8a^3-27b^3}$. 20. $\frac{x^2+2x-3}{x^2+5x+6}$. 21. $\frac{x^2-x-12}{x^2+6x+9}$.

22. $\frac{x^4+x^2-2}{x^4+5x^2+6}$. 23. $\frac{4a^2-5ab-6b^2}{8a^2+2ab-3b^2}$. 24. $\frac{ax-ab}{ax+3x-3b-ab}$.

25. $\frac{5x^2+4x-1}{5x^2+19x-4}$. 26. $\frac{x^3-ax^2+b^2x-ab^2}{x^3-ax^2-b^2x+ab^2}$.

27. $\frac{3x^2+16x-35}{5x^2+33x-14}$.

28. $\frac{x^{2n}+2x^n+1}{x^{2n}+3x^n+2}$.

29. $\frac{bx+2}{2b+(b^2-4)x-2bx^2}$.

30. $\frac{a^2-(b-c)^2}{(a+c)^2-b^2}$.

31. $\frac{(a+b)^2-4c^2}{a^2-(b+2c)^2}$.

32. $\frac{a^2+a+b-b^2}{1-(a-b)^2}$.

33. $\frac{25a^2-9(b-1)^2}{6b-10a-6}$.

34. $\frac{x^2+xz-y^2-yz}{x^2-y^2-2yz-z^2}$.

35. $\frac{1-a^2}{(1+ax)^2-(a+x)^2}$.

36. $\frac{x^5-x^4y-xy^4+y^5}{x^4-x^3y-x^2y^2+xy^3}$.

37. $\frac{3x^3-8x^2+8x-5}{2x^3+5x^2-5x+7}$.

38. $\frac{6x^3+11x^2-6x-5}{3x^3+10x^2+3x-10}$.

39. $\frac{x^3-x^2+2}{x^3-3x^2+4x-2}$.

40. $\frac{2x^3-13x^2+19x-20}{2x^3+9x^2-14x+24}$.

41. $\frac{x^3-5x^2+13x-14}{x^3-x^2+x+14}$.

42. $\frac{8x^3+2x^2-5x+1}{8x^3+10x^2-11x+2}$.

Reduction of Two or More Fractions to a Lowest Common Denominator.

11. Two or more fractions are said to have a common denominator when their denominators are the same.

E.g., $\frac{a}{b}$ and $\frac{c}{b}$; $\frac{x}{a^2-x^2}$ and $\frac{x-y}{(a+x)(a-x)}$.

The **Lowest Common Denominator** (L. C. D.) of two or more fractions is the L. C. M. of their denominators.

E.g., the L. C. D. of $\frac{a}{b^2c}$ and $\frac{d}{bc^2}$ is b^2c^2.

12. *The value of a fraction is not changed if both numerator and denominator be multiplied by the same number, not* 0.

E.g., $$\frac{a-x}{a+x}=\frac{(a-x)\times(a+x)}{(a+x)\times(a+x)}=\frac{a^2-x^2}{(a+x)^2}.$$

Let the value of the fraction $\frac{a}{b}$ be denoted by v, or

$$v = \frac{a}{b}.$$

Multiplying by b, $\quad vb = \frac{a}{b} \times b = a.$

Multiplying by n, $\quad vbn = an.$

Dividing by bn, $\quad v = an \div bn = \frac{an}{bn}.$

But $\quad v = \frac{a}{b}.$

Therefore $\quad \frac{a}{b} = \frac{an}{bn}.$

13. Ex. 1. Reduce $\frac{a}{b^2c}$ and $\frac{d}{bc^2}$ to equivalent fractions having a lowest common denominator.

Their required L. C. D. is b^2c^2.

Multiplying both terms of $\frac{a}{b^2c}$ by $b^2c^2 \div b^2c$, $=c$, we have $\frac{ac}{b^2c^2}$;

and both terms of $\frac{d}{bc^2}$ by $b^2c^2 \div bc^2$, $=b$, we have $\frac{bd}{b^2c^2}$.

Ex. 2. Reduce x, $=\frac{x}{1}$, and $\frac{y}{x-y}$ to equivalent fractions having a lowest common denominator.

The required L. C. D. is $x - y$.

Multiplying both terms of $\frac{x}{1}$ by $x - y$, we have $\frac{x^2 - xy}{x - y}$;

and both terms of $\frac{y}{x-y}$ by 1, we have, $\frac{y}{x-y}$.

Ex. 3. Reduce

$$\frac{1}{x^2 - 3x + 2}, = \frac{1}{(x-1)(x-2)},$$

and

$$\frac{2}{x^2 - 1}, = \frac{2}{(x-1)(x+1)},$$

to equivalent fractions having a lowest common denominator.

The required L. C. D. is $(x-1)(x-2)(x+1)$.

Multiplying both terms of the first fraction by

$$(x-1)(x-2)(x+1) \div (x-1)(x-2), = x+1,$$

we have $$\frac{x+1}{(x-1)(x-2)(x+1)};$$

and both terms of the second fraction by

$$(x-1)(x-2)(x+1) \div (x-1)(x+1), = x-2,$$

we have $$\frac{2x-4}{(x-1)(x-2)(x+1)}.$$

14. These examples illustrate the following method:

Take the L. C. M. of the denominators as the required denominator.

Divide this denominator by the denominator of each fraction; and multiply both numerator and denominator of the fraction by the quotient.

EXERCISES II.

Reduce the following fractions to equivalent fractions having a lowest common denominator:

1. $1, \frac{x}{4}$. **2.** $\frac{4m}{3}, \frac{5n}{6}$. **3.** $\frac{5a^2b}{14}, \frac{2ab^2}{21}$.

4. $1-a, \frac{a^2}{a+1}$. **5.** $m, \frac{1+4m}{m-4}$. **6.** $\frac{15}{14xy^2}, \frac{2x}{3y^3}$.

7. $\frac{3}{5a^2b}, 1, \frac{7}{15abx}$. **8.** $\frac{2-3x}{4x}, \frac{5+2x}{12x^2}$. **9.** $\frac{5a-4b}{6a^2b}, \frac{3b-2a}{8ab^2}$.

10. $1, \frac{1}{x-1}$. **11.** $\frac{1}{x+2}, \frac{5}{3x+6}$.

12. $\frac{1}{x^2-49}, \frac{3}{4x+28}$ **13.** $\frac{2}{x}, \frac{3}{2x-1}, \frac{2x}{4x^2-1}$.

14. $\frac{1}{x-3}, \frac{3}{x^2-9}, \frac{5}{3x+9}$. **15.** $\frac{5}{x+2}, \frac{3}{x^2+x-2}, \frac{1}{x^2-4}$.

16. $\frac{b}{ax+ab}, \frac{a}{x^2-b^2}, \frac{c}{bx-ab}$. **17.** $\frac{x}{x-1}, \frac{1}{x+1}, \frac{1}{1-x^2}$.

18. $\dfrac{m}{y(x-y)}, \dfrac{y}{m(y-x)}, \dfrac{1+m}{my}.$ **19.** $\dfrac{ax-b}{ax+ab}, \dfrac{a-bx}{bx+b^2}, \dfrac{1}{a^2b^2}.$

20. $\dfrac{a}{1-a}, \dfrac{1}{a^2-a}, \dfrac{3a+1}{a^2-1}.$ **21.** $\dfrac{3}{2x-2}, \dfrac{5}{x^2-2x+1}, \dfrac{x}{1-x^2}.$

22. $\dfrac{1}{n-m}, \dfrac{3nm}{n^3-m^3}, \dfrac{m-n}{m^2+mn+n^2}.$

23. $\dfrac{1}{x^2+2x-8}, \dfrac{1}{x^2-5x+6}, \dfrac{2}{2x^2+x-10}$

24. $\dfrac{3}{x^2+2ax-3a^2}, \dfrac{1}{x^2-9a^2}, \dfrac{4}{x^2+4ax+3a^2}.$

25. $\dfrac{1}{(a-c)(a-b)}, \dfrac{1}{(b-a)(b-c)}, \dfrac{1}{(c-a)(c-b)}.$

Equations.

15. Ex. **1.** Solve the equation $2x+\dfrac{x}{4}=9.$

Multiplying by 4, $\quad 4\times 2x+4\times\dfrac{x}{4}=4\times 9;\qquad(1)$

or, since $4\times\dfrac{x}{4}=x$, $\quad 8x+x=36.$

Uniting terms, $\quad 9x=36.$

Dividing by 9, $\quad x=4.$

The step represented by (1) is called *clearing the equation of fractions*, and should be performed mentally.

To clear of fractions, we multiplied by 4, the denominator of the fractional term. If the equation contains more than one fraction, we multiply by their L. C. D.

Ex. **2.** Solve the equation $\dfrac{x}{5}-\dfrac{2x-1}{3}=3-x.$

The L. C. D. is 15.

Multiplying by 15, $\quad 3x-5(2x-1)=15(3-x).\qquad(1)$

Removing parentheses, $\quad 3x-10x+5=45-15x.$

Transferring terms, $\quad 3x-10x+15x=45-5.$

Uniting terms, $\quad 8x=40.$

Dividing by 8, $\quad x=5.$

16. Observe that the sign of a fraction affects each term of the numerator, or the dividing line between the numerator and the denominator has the same effect as parentheses.

E.g.,
$$-\frac{a-b+c}{d} = -(a-b+c) \div d$$
$$= (-a+b-c) \div d$$
$$= \frac{-a+b-c}{d}.$$

Thus, in Ex. 2, Art. 15, the sign $-$ before the fraction $\frac{2x-1}{3}$ changes the signs of *both* terms in its numerator, and not simply the sign of the first term, when the denominator is removed. This caution should be kept in mind, and step (1) omitted in clearing of fractions.

17. Ex. Solve the equation $\frac{x+1}{6} - \frac{x-1}{8} = \frac{x+2}{12}.$

The L. C. D. is 24.

Multiplying by 24, $4x + 4 - 3x + 3 = 2x + 4.$

Transferring terms, $4x - 3x - 2x = -3.$

Uniting terms, $-x = -3.$

Dividing by -1, $x = 3.$

EXERCISES III.

Solve each of the following equations:

1. $x + \frac{x}{2} = 18.$
2. $x - \frac{3x}{8} = 5.$
3. $\frac{7x}{10} + 6 = x.$
4. $\frac{x}{2} + \frac{x}{4} = 15.$
5. $\frac{2x}{3} - \frac{x}{2} = 5.$
6. $\frac{3x}{5} - \frac{x}{2} = 5.$
7. $\frac{x-2}{3} = \frac{3-x}{2}.$
8. $\frac{x-4}{5} = \frac{5-x}{4}.$
9. $\frac{3x-2}{4} = \frac{3x+2}{5}.$
10. $\frac{x+3}{2} + \frac{x}{3} = 4.$
11. $\frac{x-1}{4} + \frac{x}{5} = 2.$
12. $\frac{3x-2}{5} - \frac{x}{4} = 1.$
13. $\frac{6x-5}{4} - \frac{5x}{3} = -3.$
14. $\frac{3x}{4} - \frac{x+4}{6} = 4.$
15. $\frac{5x}{8} - \frac{x-10}{6} = 2.$

16. $\dfrac{8x+6}{3}-\dfrac{5x-1}{2}=3.$ **17.** $\dfrac{5-x}{12}-\dfrac{x-4}{15}=\dfrac{x-3}{20}.$

18. $\dfrac{x-6}{2}-\dfrac{x-3}{12}=\dfrac{1}{3}-\dfrac{x+1}{16}.$

19. $\dfrac{5-x}{4}=\dfrac{x-3}{6}+\dfrac{x-1}{16}-\dfrac{7}{12}.$

20. $\dfrac{x-4}{6}-\dfrac{x-3}{8}=\dfrac{1}{3}-\dfrac{x-2}{12}.$

21. $\dfrac{x-5}{18}-\dfrac{x-4}{20}+\dfrac{x+2}{24}=\dfrac{x-12}{3}.$

22. $\dfrac{x-9}{6}-\dfrac{x-1}{18}=\dfrac{x-5}{16}-\dfrac{x-3}{20}.$

Problems.

18. Pr. In a number of two digits, the units' digit is two-thirds of the tens' digit. If the digits be interchanged, the resulting number will be 18 less than the given number. What is the number?

Let x stand for the tens' digit;

then $\frac{2}{3}x$ stands for the units' digit.

The given number is $10x+\frac{2}{3}x$,

and the resulting number is $10\times\frac{2}{3}x+x, =\frac{20}{3}x+x.$

The problem states,

in *verbal* language: *the given number minus the resulting number is* 18;

in *algebraic* language: $10x+\frac{2}{3}x-(\frac{20}{3}x+x)=18.$

Removing parentheses, $10x+\frac{2}{3}x-\frac{20}{3}x-x=18.$

Clearing of fractions, $30x+2x-20x-3x=54.$

Uniting terms, $9x=54.$

Dividing by 9, $x=6,$

the tens' digit.

Then the units' digit is $\frac{2}{3}x, =4.$

Therefore the required number is 64.

EXERCISES IV.

1. A son is $\frac{1}{4}$ as old as his father, and in 18 years he will be $\frac{1}{2}$ as old. How old is each?

2. A son is $\frac{1}{3}$ as old as his father, and 6 years ago he was $\frac{1}{6}$ as old. How old is each?

3. A boy lost $\frac{1}{5}$ of his money, and afterward found 12 cents. He then had twice as much as at first. How much money had he at first?

4. Two men invest equal amounts. The first one loses \$600, and the second one gains \$600. The first then has only $\frac{8}{9}$ as much as the second. How much did each invest?

5. Divide 65 into 3 parts, so that the second shall be 8 greater than the first, and the third $\frac{3}{2}$ of the sum of the first and second.

6. From a cask full of water, $\frac{1}{3}$ of the contents is drawn off. If 10 gallons are then poured into it, it will contain $\frac{7}{9}$ of its original contents. What is the capacity of the cask?

7. The sum of the two digits of a number is 12. If the digits be interchanged, the resulting number will exceed the original one by three-fourths of the original number. What is the number?

8. A merchant paid 30 cents a yard for a piece of cloth. He sold one-half for 35 cents a yard, one-third for 29 cents a yard, and the remainder for 32 cents a yard, gaining \$18.15 by the transaction. How many yards did he buy?

9. A woman sells $\frac{1}{2}$ of an apple more than one-half of her apples. She next sells $\frac{1}{2}$ of an apple more than one-half of the apples not yet sold, and then has 6 apples left. How many apples had she at first?

10. A merchant lost $\frac{1}{5}$ of his capital, and then $\frac{1}{4}$ of what remained. If he then had \$12,000 capital, how much had he at first?

11. Thirteen coins, dollars and quarter-dollars, amount to \$9.25. How many coins of each kind are there?

12. A box contains a number of pencils, of which $\frac{2}{3}$ are red, $\frac{1}{4}$ are blue, and 3 are black. How many pencils are red, and how many are blue?

13. The deposits in a bank during three days amounted to \$77,700. If the deposits each day after the first were $\frac{3}{4}$ of those of the preceding day, how many dollars were deposited each day?

14. A father leaves his property to his three sons as follows: to the first, \$3000 less than $\frac{1}{2}$ of his property; to the second, \$2400 less than $\frac{1}{3}$ of his property; and to the third, \$1800 less than $\frac{1}{4}$ of his property. What is the amount of his property?

15. A father divided his property equally among his sons. To the oldest son he gave \$300 and $\frac{1}{10}$ of what remained; to the second son he gave \$600 and $\frac{1}{10}$ of what was then left; to the third son he gave \$900 and $\frac{1}{10}$ of the remainder; and so on. What was the amount of his property, and how many sons had he?

16. The height of the first platform of the Eiffel Tower is 8 meters more than $\frac{1}{6}$ of the whole height; the second platform is twice as high as the first, and 160 meters less than the third; the third is 1 meter greater than $\frac{11}{12}$ of the entire height. What is the height of the tower, and of each platform?

17. Jupiter has 1 more moon than Uranus, and Uranus half as many moons as Saturn; Mars has 3 less than Jupiter, and Neptune half as many as Mars. If these planets together have 20 moons, how many has each?

18. A leaves a certain town P, travelling at the rate of 21 miles in 5 hours; B leaves the same town 3 hours later and travels in the same direction at the rate of 21 miles in 4 hours. After how many hours will B overtake A, and at what distance from P?

19. A train runs from A to B at the rate of 30 miles an hour; and returning runs from B to A at the rate of 28 miles

an hour. The time required to go from A to B and return is 15 hours, including 30 minutes' stop at B. How far is A from B?

20. A servant is to receive \$170 and a dress for one year's services. At the end of 7 months she leaves her place and receives \$95 and the dress. What is the value of the dress?

21. A cistern has three pipes which can empty it in 6, 8, and 10 hours respectively. After all three pipes have been open for 2 hours they have discharged 94 gallons. What is the capacity of the cistern?

22. A wall can be built by 20 workmen in 11 days, or by 30 other workmen in 7 days. If 22 of the first class work together with 21 of the second class, after how many days will the work be completed?

23. At 6 o'clock the hands of a clock are in a straight line. At what time between 7 and 8 o'clock will they be again in a straight line? At what time between 9 and 10 o'clock?

Addition and Subtraction of Fractions.

19. Add $\frac{b}{c}$ to $\frac{a}{c}$. We have

$$\frac{a}{c}+\frac{b}{c}=a \div c + b \div c = (a+b) \div c = \frac{a+b}{c}.$$

This proves the following method of adding two or more fractions which have a common denominator:

The numerator of the sum is the sum of the numerators, and the denominator is the common denominator.

A similar method is applied in subtracting fractions.

E.g.,
$$\frac{2x}{x-1}-\frac{1+x}{x-1}=\frac{2x-(1+x)}{x-1}=\frac{x-1}{x-1}=1.$$

20. If the fractions to be added or subtracted do not have a common denominator, they should first be reduced to equivalent fractions having a lowest common denominator.

Ex. 1. Simplify $\frac{a}{b^2c}+\frac{d}{bc^2}$.

We have $$\frac{a}{b^2c}+\frac{d}{bc^2}=\frac{ac}{b^2c^2}+\frac{bd}{b^2c^2}=\frac{ac+bd}{b^2c^2}.$$

Ex. 2. Simplify $\frac{2x-5y}{5}-\frac{3x-6y+2z}{4}$.

Reducing to L. C. D., we have

$$\frac{8x-20y}{20}-\frac{15x-30y+10z}{20}$$

$$=\frac{8x-20y-(15x-30y+10z)}{20}$$

$$=\frac{8x-20y-15x+30y-10z}{20}=\frac{-7x+10y-10z}{20}.$$

The expressions in this example are not *algebraic* fractions.

The beginner should be careful in subtracting a fraction to *change the sign of each term of the numerator*, and not that of the first term only.

In like manner we may change the sign of each term of the numerator (or denominator), if we change the sign of the fraction. Thus, in the result of Ex. 2, we have

$$\frac{-7x+10y-10z}{20}=-\frac{7x-10y+10z}{20}.$$

Ex. 3. Simplify $\frac{1}{x-1}-\frac{2}{x+1}+\frac{3x}{x^2-1}$.

The L. C. D. is x^2-1.

Therefore, $$\frac{1}{x-1}-\frac{2}{x+1}+\frac{3x}{x^2-1}=\frac{x+1}{x^2-1}-\frac{2x-2}{x^2-1}+\frac{3x}{x^2-1}$$

$$=\frac{x+1-2x+2+3x}{x^2-1}$$

$$=\frac{2x+3}{x^2-1}.$$

EXERCISES V.

Simplify the following expressions:

1. $\frac{a}{b}+\frac{b}{a}$.
2. $\frac{a}{8}+\frac{5a}{16}$.
3. $\frac{5}{4n}-\frac{7}{6n}$.
4. $\frac{1}{ab}+\frac{1}{ac}$.
5. $\frac{1}{xy}+\frac{1}{xz}-\frac{1}{yz}$.
6. $\frac{b}{2a}+\frac{3b}{4a}-\frac{5b}{6a}$.
7. $\frac{1}{ab^2}+\frac{1}{a^2b}-\frac{1}{a^2b^2}$.
8. $\frac{3x+5}{3}+\frac{3x-1}{2}$.
9. $\frac{3z+5y}{6}-\frac{2z+3y}{4}$.
10. $\frac{3x-2}{5}-\frac{x+7}{2}+4$.
11. $\frac{a-3}{2}-\frac{a-5}{6}-\frac{4-a}{8}$.
12. $\frac{x-1}{2}-\frac{x-2}{3}+\frac{x+7}{6}$.
13. $\frac{3-2a}{3}+\frac{3a-2}{5}-\frac{6a+2}{10}$.
14. $\frac{5-3x}{4}-\frac{5x-4}{10}-\frac{25-19x}{15}$.
15. $\frac{x-2}{3x}-\frac{2x-5}{4x^2}+\frac{4-3x}{9x}$.
16. $\frac{2x-4y}{5}-\frac{5x+2y-3z}{10}+\frac{x+16y-5z}{15}$.
17. $\frac{x-y-z}{4}-\frac{5y-3z-x}{7}-\frac{5z-10y+6x}{14}$.
18. $\frac{x^2-3x+1}{18}-\frac{3x^2-2x-4}{12}-\frac{6x-3x^2}{16}$.
19. $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}-\frac{a-4b}{21}$.
20. $\frac{1}{x-3}-\frac{1}{x+4}$.
21. $\frac{x}{x-1}+\frac{1}{2x-1}$.
22. $\frac{x-1}{x-2}-\frac{x-3}{x-1}$.
23. $\frac{x-1}{x+1}-\frac{x-2}{x+2}$.
24. $\frac{m+n}{m-n}-\frac{m-n}{m+n}$.
25. $\frac{n}{a^n+1}-\frac{n}{a^n-1}$.
26. $\frac{2a}{a^2-1}-\frac{1}{a+1}$.
27. $\frac{ac}{a^2-4y^2}+\frac{bd}{ac+2cy}$.
28. $\frac{1}{1+x}+\frac{1}{1-x}-\frac{2x}{1-x^2}$.
29. $\frac{3a-1}{a^2-9}-\frac{1-3a}{a+3}-\frac{3a-16}{a-3}$.

30. $\dfrac{3a}{a+x}+\dfrac{a}{a-x}-\dfrac{2ax}{a^2-x^2}.$ 31. $\dfrac{x-1}{6x+24}-\dfrac{1-x}{x^2-16}-\dfrac{x-5}{3x-12}.$

32. $\dfrac{2m-3}{1-4m^2}+\dfrac{3}{1-2m}+\dfrac{2}{m}.$ 33. $\dfrac{2}{x}+\dfrac{x-6}{3x+6}-\dfrac{1}{x^2+2x}.$

34. $\dfrac{5a}{9a^2-25b^2}-\dfrac{2a+3b}{6ad+10bd}-\dfrac{4a-b}{6ad-10bd}.$

35. $\dfrac{2}{x^2-3x+2}-\dfrac{3}{x^2-5x+6}+\dfrac{4}{x^2-4x+3}.$

36. $\dfrac{4x}{x^2-3ax+2a^2}-\dfrac{3x}{2x^2-3ax+a^2}-\dfrac{5x}{2x^2-5ax+2a^2}.$

37. $\dfrac{1}{a-1}-\dfrac{a^2+2a}{a^3-1}.$ 38. $\dfrac{1}{x+1}+\dfrac{x^2+x}{x^3+1}.$

39. $\dfrac{a-2n}{a^3+n^3}-\dfrac{a-n}{a^2n-an^2+n^3}-\dfrac{1}{an+n^2}.$

40. $\dfrac{1}{n-m}-\dfrac{3nm}{n^3-m^3}-\dfrac{m-n}{m^2+mn+n^2}.$

41. $\dfrac{5}{x-2}+\dfrac{7}{x-1}-\dfrac{5}{x+2}-\dfrac{7}{x+1}.$

42. $\dfrac{4}{x+7}-\dfrac{1}{x-8}-\dfrac{4}{x-7}+\dfrac{1}{x+8}.$

21. Frequently the denominators are multinomials in the same letter of arrangement, but not arranged to the same order of powers.

Ex. 1. Simplify $\dfrac{x}{x-1}+\dfrac{2x}{1-x^2}-\dfrac{2x}{x+1}.$

It is better first to change the second fraction so that the denominators shall be arranged in the same order. We then have, by Art. 7 (ii.),

$$\frac{x}{x-1}-\frac{2x}{x^2-1}-\frac{2x}{x+1}.$$

The L. C. D. is x^2-1.

Therefore,

$$\frac{x}{x-1}-\frac{2x}{x^2-1}-\frac{2x}{x+1}=\frac{x(x+1)-2x-2x(x-1)}{x^2-1}$$
$$=\frac{x^2+x-2x-2x^2+2x}{x^2-1}$$
$$=\frac{x-x^2}{x^2-1}=-\frac{x(x-1)}{x^2-1}=-\frac{x}{x+1}.$$

As in this example, the result of the addition should be reduced to its lowest terms.

Ex. 2. Simplify $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}+\dfrac{1}{(c-a)(c-b)}$

Changing the fractions into equivalent fractions, whose denominators, taken in pairs, have one common factor, we have

$$\frac{1}{(a-b)(a-c)}-\frac{1}{(a-b)(b-c)}+\frac{1}{(a-c)(b-c)}$$
$$=\frac{b-c}{(a-b)(a-c)(b-c)}-\frac{a-c}{(a-b)(b-c)(a-c)}$$
$$+\frac{a-b}{(a-c)(b-c)(a-b)}=\frac{b-c-a+c+a-b}{(a-b)(a-c)(b-c)}=0.$$

EXERCISES VI.

Simplify the following expressions:

1. $\dfrac{3}{x^2-4}-\dfrac{6}{2-x}.$

2. $\dfrac{a}{3-a}-\dfrac{9}{a^2-3a}.$

3. $\dfrac{b}{a^2-ab}-\dfrac{a}{b^2-ab}.$

4. $\dfrac{x+4}{5x-10}-\dfrac{x-2}{6-3x}.$

5. $\dfrac{3}{2x-1}+\dfrac{7}{2x+1}-\dfrac{4-20x}{1-4x^2}.$

6. $\dfrac{m}{m-n}+\dfrac{2mn}{n^2-m^2}-\dfrac{2m}{m+n}.$

7. $\dfrac{a-1}{a+1}-\left(\dfrac{a+1}{1-a}+\dfrac{a^2+1}{a^2-1}\right).$

8. $\dfrac{1}{x^2-3x+2}-\dfrac{1}{1-x^2}.$

9. $\dfrac{1}{x^4+x^2+1}-\dfrac{1}{x-1-x^2}+\dfrac{1}{x+1+x^2}.$

10. $\dfrac{1}{2x^2-4x+2}+\dfrac{1}{2x^2+4x+2}-\dfrac{1}{1-x^2}.$

11. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}.$

12. $\dfrac{ab}{(b-c)(c-a)}+\dfrac{bc}{(c-a)(a-b)}+\dfrac{ca}{(a-b)(b-c)}.$

13. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}+\dfrac{1}{c(c-a)(c-b)}.$

14. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}.$

15. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}.$

16. $\dfrac{bc}{a(a^2-b^2)(a^2-c^2)}+\dfrac{ac}{b(b^2-a^2)(b^2-c^2)}+\dfrac{ab}{c(c^2-a^2)(c^2-b^2)}.$

17. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ac}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}.$

Reduction of Mixed Expressions to Improper Fractions.

22. A **Proper Fraction** is one whose numerator is of lower degree than its denominator in a common letter of arrangement.

E.g., $$\frac{1}{x+1},\ \frac{x-2}{x^2+2x-1}.$$

An **Improper Fraction** is one whose numerator is of the same or of a higher degree than its denominator in a common letter of arrangement.

E.g., $$\frac{x}{x+1},\ \frac{x^3+3x^2+x-1}{x^2+2x-1}.$$

If both integral and fractional terms occur in an expression, it is sometimes called a **Mixed Expression.**

23. Ex. 1. Reduce $a+\frac{b}{c}$ to an improper fraction. First reducing a to the form of a fraction with denominator c, we have

$$a+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}.$$

This example illustrates the following method:

Multiply the integral part by the denominator of the fractional part. To this product add algebraically the numerator of the fractional part, and write the sum as the required numerator.

Ex. **2.** Simplify $1 - x + x^2 - \dfrac{x^3}{1+x}$.

We have $1 - x + x^2 - \dfrac{x^3}{1+x} = \dfrac{(1 - x + x^2)(1+x) - x^3}{1+x}$

$$= \frac{1 + x^3 - x^3}{1+x} = \frac{1}{1+x}.$$

EXERCISES VII.

Simplify the following expressions:

1. $2a - \dfrac{a}{3}$. **2.** $7 + \dfrac{1}{a}$. **3.** $m + \dfrac{1}{m}$.

4. $\dfrac{a-x}{x} + 1$ **5.** $1 + \dfrac{1}{x-1}$. **6.** $3a + \dfrac{1-8a}{3}$.

7. $2m - \dfrac{3m-5n}{4}$. **8.** $a - \dfrac{a^2}{a+b}$. **9.** $4 + \dfrac{8a}{2a+3b}$.

10. $x - \dfrac{3x-4}{3-x}$. **11.** $1 + \dfrac{(a-b)^2}{4ab}$.

12. $x + 4 - \dfrac{9x+20}{x+5}$. **13.** $5x - 6 - \dfrac{42-x}{x-7}$.

14. $a + b - \dfrac{a^2}{a-b}$. **15.** $a - b + \dfrac{4ab}{a-b}$.

16. $1 - \left(a - \dfrac{a^2}{1+a}\right)$. **17.** $a^2 + ax + x^2 + \dfrac{x^3}{a-x}$.

Reduction of Improper Fractions to Mixed Expressions.

24. Ex. **1.** Reduce $\dfrac{2x^2 + x + 5}{x+1}$ to a mixed expression.

We have

$$\begin{array}{r|l} 2x^2 + x + 5 & x+1 \\ \underline{2x^2 + 2x\phantom{{}+5}} & 2x-1 \\ -x + 5 & \\ \underline{-x - 1} & \\ 6 & \end{array}$$

But by Ch. III., Art. 47, we have

$$(2x^2+x+5)\div(x+1)=2x-1+6\div(x+1),$$

or
$$\frac{2x^2+x+5}{x+1}=2x-1+\frac{6}{x+1}.$$

This example illustrates the following method:

Divide the numerator by the denominator, until the remainder is of lower degree than the divisor.

Write the remainder as the numerator of a fraction whose denominator is the divisor.

Add this fraction to the integral part of the quotient.

Ex. 2. Reduce $\dfrac{x^3+x^2-4x+3}{x^2+2x-1}$ to a mixed expression.

We have

$$\begin{array}{r|l}
x^3+x^2-4x+3 & x^2+2x-1 \\
x^3+2x^2-x & x-1 \\
\hline
-x^2-3x+3 & \\
-x^2-2x+1 & \\
\hline
-x+2 &
\end{array}$$

Therefore,
$$\frac{x^3+x^2-4x+3}{x^2+2x-1}=x-1+\frac{-x+2}{x^2+2x-1}$$
$$=x-1-\frac{x-2}{x^2+2x-1}.$$

EXERCISES VIII.

Reduce each of the following fractions to equivalent fractional expressions, containing only proper fractions:

1. $\dfrac{x^3+x^2-1}{x^2}$.
2. $\dfrac{x^2-x-1}{x^2}$.
3. $\dfrac{10a^2-3a+4}{5a^2}$.
4. $\dfrac{6a^3-9a^2b+5b}{3a}$.
5. $\dfrac{x^2+x-xy}{x-y}$.
6. $\dfrac{a^2-b^2-a}{a-b}$.
7. $\dfrac{9x^2-9x+3}{x-1}$.
8. $\dfrac{2x^2+x-5}{x+1}$.
9. $\dfrac{21x^2+20x-1}{3x+2}$.
10. $\dfrac{m^3-n^3-1}{m-n}$.
11. $\dfrac{x^3-3x^2+2x-3}{x-1}$.

12 $\dfrac{m^3 - mn^2 - m^2n + n^3 + 1}{m - n}$.

13. $\dfrac{5x^3 - 3x - 14}{x^2 - 2}$.

14. $\dfrac{4x^3 + 21x + 9}{x^2 + 7}$.

15. $\dfrac{x^3 + x^2 - 2}{x^2 - 1}$.

Multiplication of Fractions.

25. Multiply $\frac{a}{b}$ by $\frac{c}{d}$. Let the value of $\frac{a}{b}$ be denoted by v, and that of $\frac{c}{d}$ by w; or

$$v = \frac{a}{b}, \text{ and } w = \frac{c}{d}.$$

Multiplying the first equation by b, and the second by d, we have

$$vb = a, \text{ and } wd = c.$$

Multiplying together corresponding members of these equations, we have

$$vb \times wd = ac,$$

or

$$vw \times bd = ac.$$

Dividing by bd,

$$vw = ac \div bd = \frac{ac}{bd}.$$

But

$$vw = \frac{a}{b} \times \frac{c}{d}.$$

Therefore

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

This proves the following method of multiplying fractions:

The numerator of the product is the product of the numerators; and the denominator of the product is the product of the denominators.

26. Ex. 1. Find the product $\dfrac{15a^3b^2}{22x^2y^5} \times \dfrac{14xy^2}{25a^2b}$.

The factor 5 is common to the numerator of the first fraction and the denominator of the second. Since to cancel a common factor *before* multiplication is equivalent to cancelling it *after* the multiplication, we should first cancel 5. For a similar

reason we should cancel the factors, 2, a^2, b, x, and y^2 before the multiplication. We then have

$$\frac{3\,ab}{11\,xy^3} \times \frac{7}{5} = \frac{21\,ab}{55\,xy^3}.$$

In general, if the numerator of one fraction and the denominator of another have common factors, such factors should be cancelled before the multiplications are performed.

Ex. 2. Find the product $\dfrac{8\,a^2}{a^2 - b^2} \times \dfrac{(a+b)^2}{4\,a}$.

Cancelling the common factors, $4\,a$ and $a + b$, we have

$$\frac{2\,a}{a-b} \times \frac{a+b}{1} = \frac{2\,a(a+b)}{a-b}.$$

27. Ex. Find the product $\dfrac{x-y}{x^2+y^2} \times (x+y)$.

We have $$\frac{x-y}{x^2+y^2} \times \frac{x+y}{1} = \frac{x^2-y^2}{x^2+y^2}.$$

Observe that *a fraction is multiplied by an integer, by multiplying its numerator by the integer.*

28. If one of the factors is a mixed expression, it should first be reduced to an improper fraction.

Ex. Find the product $\left(1 - \dfrac{1}{x}\right)\left(\dfrac{1}{x^2-1}\right)$.

We have $$\left(1 - \frac{1}{x}\right)\left(\frac{1}{x^2-1}\right) = \frac{x-1}{x} \times \frac{1}{x^2-1}$$
$$= \frac{1}{x} \times \frac{1}{x+1} = \frac{1}{x(x+1)}.$$

EXERCISES IX.

Multiply:

1. $\dfrac{4}{x} \times 3$.
2. $\dfrac{a}{5\,x} \times 5$.
3. $\dfrac{x}{15\,y} \times 10\,y$.
4. $\dfrac{5\,x}{3\,y^2} \times 12\,y^3$.
5. $\dfrac{6\,x}{7\,z} \times \dfrac{7\,z}{9\,y}$.
6. $\dfrac{2\,a^2b}{5\,x^2} \times \dfrac{10\,x}{ay}$.

7. $\frac{7\,bx}{3\,a^3}\times\frac{15\,ab^3}{14\,x^2y}$. 8. $\frac{15\,a^3b^2}{25\,x^2y^5}\times\frac{14\,xy^2}{25\,a^2b}$. 9. $\frac{16\,x^2y}{21\,a^2b}\times\frac{3\,a^3b^2}{4\,x^4y^2}$.

10. $\frac{4\,a^2}{5\,b^3}\times\frac{15\,b}{8\,c}\times\frac{2\,bc}{3\,a}$. 11. $\frac{x}{2\,b^2c^2}\times\frac{3\,bcy}{ax^3}\times\frac{4\,ab}{9\,xy^2}$.

12. $\frac{5\,x}{15\,a-10\,b}\times(3\,a-2\,b)$. 13. $\frac{8\,a^2}{a^2-b^2}\times\frac{a+b}{2\,a}$.

14. $\frac{ab^2-b^3}{a^2+ab}\times\frac{a^3-ab^2}{2\,b^2}$. 15. $\frac{x+y}{6\,x-12\,y}\times\frac{x^2-4\,y^2}{(x+y)^2}$.

16. $\frac{a^2b+ab^2}{a^3b+ab^3}\times\frac{a^4-b^4}{5\,ab\,(a+b)^2}$. 17. $\frac{5\,x+6\,y}{x^2+6\,x+9}\times\frac{x^2-9}{25\,x^2-36\,y^2}$.

18. $\frac{x-3}{x+1}\times\frac{x^2+2\,x+1}{x^3-27}$. 19. $\frac{x^3-1}{x+4}\times\frac{x^2+8\,x+16}{x^2+x+1}$.

20. $\frac{a\,(a+b)}{a^2-2\,ab+b^2}\times\frac{b\,(a-b)}{a^2+2\,ab+b^2}$.

21. $\frac{6\,ax-15\,bx}{40\,ay+15\,dy}\times\frac{8\,ax+3\,dx}{4\,a^2-25\,b^2}$.

22. $\frac{x^4-y^4}{(x+y)^2}\times\frac{x^2-y^2}{x^2+y^2}\times\frac{x+y}{(x-y)^2}$.

23. $\frac{x^4-y^4}{a^3+b^3}\times\frac{a^2-ab+b^2}{x-y}\times\frac{a+b}{x+y}$.

24. $\frac{x^2-(a+b)\,x+ab}{x^2-(a+c)\,x+ac}\times\frac{x^2-c^2}{x^2-b^2}$.

25. $\frac{a^2-(b-c)^2}{x^2-y^2}\times\frac{(x+y)^2}{(a-b)^2-c^2}$.

26. $\frac{x^3-8\,y^3}{x^2-y^2}\times\frac{x+y}{x^2+2\,xy+4\,y^2}$.

27. $\frac{x^2-4}{x^2-8\,x+15}\times\frac{x^2-9}{x^2-8\,x+12}$.

28. $\frac{x-y+z}{x+y-z}\times\frac{x^2+2\,xy+y^2-z^2}{x^2-2\,xy+y^2-z^2}$.

29. $\frac{4\,x^2-9\,y^2}{22\,a^2-10\,ab}\times\frac{33\,ab-15\,b^2}{6\,ax-9\,ay}\times\frac{12\,a^2}{10\,bx+15\,by}$.

30. $\dfrac{x^2+x-6}{x^2-x-20}\times\dfrac{x^2+x-12}{x^2+x-6}\times\dfrac{x^2-3x-10}{x^2-4}.$

31. $\dfrac{y+x}{(m+n)^3}\times\dfrac{x^2-y^2}{12}\times\dfrac{(m+n)^2}{m-n}\times\dfrac{6(m^2-n^2)}{x+y}.$

32. $(x^2-x+1)\left(\dfrac{1}{x^2}+\dfrac{1}{x}+1\right).$ 33. $\left(\dfrac{a}{b}+1+\dfrac{b}{a}\right)\left(\dfrac{a}{b}-1+\dfrac{b}{a}\right).$

34. $\left(\dfrac{a+x}{a}-\dfrac{x-y}{x}\right)\times\dfrac{a^2}{x^2+ay}.$ 35. $\left(\dfrac{x^2+1}{2x-1}-\dfrac{1}{2}x\right)\times\dfrac{1-2x}{x+2}.$

36. $\dfrac{1-x^2}{1+y}\times\dfrac{1-y^2}{x+x^2}\times\left(1+\dfrac{x}{1-x}\right).$

37. $\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}\right)\times\dfrac{x+y}{2y}.$

38. $\left[a^2-ab+b^2-\dfrac{a^3-b^3}{a+b}\right]\left[1+a-\dfrac{a(b-1)}{b}\right].$

39. $\dfrac{m^2-(b-a)^2}{m^2-(a-b)^2}\times\dfrac{(m-a)^2-b^2}{(m-b)^2-a^2}\times\dfrac{am-ab+a^2}{bm-ab+b^2}.$

40. $\left(1+\dfrac{2z}{x+y-z}\right)\times\dfrac{(x+y)^2-z^2}{(x+y+z)^2}.$

41. $\dfrac{(a+b)^2-4c^2}{(a-c)^2-ab-c^2}\times\dfrac{a(a+b+1)}{(a+2c)^2-b^2}\times\dfrac{(a-b)^2-4c^2}{(a+b)^2-1}.$

Reciprocal Fractions.

29. The **Reciprocal** of a fraction is a fraction whose numerator is the denominator, and whose denominator is the numerator, of the given fraction.

E.g., the reciprocal of $\dfrac{a}{b}$ is $\dfrac{b}{a}.$

30. The product of a fraction and its reciprocal is 1.

For $$\frac{a}{b}\times\frac{b}{a}=\frac{ab}{ba}=1.$$

Division of Fractions.

31. Divide $\frac{a}{b}$ by $\frac{c}{d}$. Let the value of $\frac{a}{b}$ be denoted by v, and that of $\frac{c}{d}$ by w; or,

$$v=\frac{a}{b}, \text{ and } w=\frac{c}{d}.$$

Multiplying the first fraction by b, and the second by d, we have

$$vb=a, \text{ and } wd=c.$$

Dividing the members of the first equation by the corresponding members of the second, we have

$$vb \div wd = a \div c, \text{ or } \frac{vb}{wd}=\frac{a}{c}.$$

Multiplying by $\frac{d}{b}$, $\quad \frac{vb}{wd}\times\frac{d}{b}=\frac{a}{c}\times\frac{d}{b},$

or $$\frac{v}{w}=\frac{ad}{bc}.$$

But $$\frac{v}{w}=\frac{a}{b}\div\frac{c}{d}.$$

Therefore $$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}=\frac{a}{b}\times\frac{d}{c}.$$

This proves the following method of dividing one fraction by another:

Multiply the dividend by the reciprocal of the divisor.

32. Ex. 1. $$\frac{4(a^2-ab)}{(a+b)^2}\div\frac{6a}{a^2-b^2}=\frac{4a(a-b)}{(a+b)^2}\times\frac{(a-b)(a+b)}{6a}$$
$$=\frac{2(a-b)^2}{3(a+b)}.$$

Ex. 2. $$\frac{x^2-y^2}{x^2+y^2}\div(x-y)=\frac{x^2-y^2}{x^2+y^2}\div\frac{x-y}{1}$$
$$=\frac{x^2-y^2}{x^2+y^2}\times\frac{1}{x-y}=\frac{x+y}{x^2+y^2}.$$

Observe that *a fraction is divided by an integer by dividing its numerator, or multiplying its denominator, by the integer.*

EXERCISES X.

Simplify the following expressions:

1. $\dfrac{9}{x} \div 3.$ 2. $\dfrac{9}{a} \div 4.$ 3. $6a \div \dfrac{a}{6}.$ 4. $2xy \div \dfrac{2x}{y}.$

5. $\dfrac{3ax}{5by} \div \dfrac{9ax}{4by}.$ 6. $\dfrac{6a^2}{5y^2} \div \dfrac{3a}{15y}.$ 7. $\dfrac{4a^2b}{21x^2y^2} \div \dfrac{6ab^2}{35xy^3}.$

8. $\dfrac{a^5b^6}{x^7y^8} \div \dfrac{a^3b^4}{x^5y^6}.$ 9. $\dfrac{27a^3b^4}{16x^5y^2} \div \dfrac{9a^5b^2}{4x^3y^6}.$ 10. $\dfrac{12x^5y^6}{35a^7b^3} \div \dfrac{18x^6y^5}{7a^4b^6}.$

11. $\dfrac{x^2+7x+12}{x^2+2x-15} \div \dfrac{x+4}{x+5}.$ 12. $\dfrac{x^2-6x+8}{x^2+2x+1} \div \dfrac{x-4}{x+1}.$

13. $\dfrac{2a^3-2ab^2}{a+2b} \div \dfrac{a^2-b^2}{2a+4b}.$ 14. $\dfrac{6(a^2-b^2)^2}{7(x^3-1)} \div \dfrac{3(a+b)}{(1-x)}.$

15. $\dfrac{a^2-(b-c)^2}{(a^2-b^2)^2} \div \dfrac{a-b+c}{a^4-b^4}.$ 16. $\dfrac{x^3-1}{x^2-a^2} \div \dfrac{x^2+x+1}{x-a}.$

17. $\dfrac{a^2+ab}{a^2+b^2} \div \dfrac{a^3b+ab^3+2a^2b^2}{a^4-b^4}.$ 18. $\dfrac{1+n-n^3-n^4}{1-a^2} \div \dfrac{n^2-1}{a^2-1}.$

19. $\dfrac{1-2x}{1-x^3} \div \dfrac{1-2x+x^2-2x^3}{1+2x+2x^2+x^3}.$

20. $\dfrac{x^2+y^2-2xy-z^2}{a^2-9+4b^2+4ab} \div \dfrac{x-y+z}{a+2b-3}.$

21. $\dfrac{1-x}{x^3+x^4-x^5} \div \dfrac{1-x^3}{x^5-x^3-2x^2-x}.$

22. $\dfrac{(a+2b)a^3-(2a+b)b^3}{a^4b^4} \div \dfrac{(a+b)^2}{a^4b^2+a^2b^4}.$

23. $\dfrac{x^2+2x-3}{x^2-2x-3} \div \dfrac{x^2+4x+3}{x^2-4x+3} \times \dfrac{x^3+1}{x^3-1}.$

24. $\dfrac{x^4+x^2y^2+y^4}{x^2+y^2} \times \dfrac{x^2+y(2x+y)}{x^3-y^3} \div \dfrac{x^3+y^3}{x^2-y(2x-y)}.$

25. $\dfrac{(x+m)^2-(y+n)^2}{(x+y)^2-(m+n)^2} \div \dfrac{(x-y)^2-(n-m)^2}{(x-m)^2-(n-y)^2}.$

Complex Fractions.

33. A **Complex Fraction** is a fraction whose numerator and denominator, either or both, are fractions.

E.g.,
$$\frac{\frac{2}{3}}{\frac{4}{5}},\quad \frac{\frac{a+x}{a-x}}{\frac{a+y}{a-y}},\quad \frac{1+\frac{1}{x}}{1-\frac{1}{x}}.$$

Observe that the line which separates the terms of the complex fraction is drawn heavier than the lines which separate the terms of the fractions in its numerator and denominator.

34. Ex. 1. Simplify $\dfrac{\frac{1-x^2}{x}}{1-x}$.

Multiplying both numerator and denominator by x, we obtain

$$\frac{\frac{x(1-x^2)}{x}}{x(1-x)}=\frac{1-x^2}{x(1-x)}=\frac{1+x}{x}.$$

To reduce a complex fraction to a simple fraction:

Multiply both its terms by the L. C. D. of the fractions in the numerator and denominator.

Ex. 2.
$$\frac{3}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}=\frac{3}{x+\dfrac{1}{\dfrac{4}{3-x}}}=\frac{3}{x+\dfrac{3-x}{4}}$$
$$=\frac{3}{\dfrac{3x+3}{4}}=\frac{4}{x+1}.$$

Observe that in this reduction the work proceeds from below upward.

EXERCISES XI.

Simplify the following expressions:

1. $\dfrac{a+\dfrac{a^2}{c}}{b+\dfrac{bc}{a}}$.

2. $\dfrac{a-\dfrac{ax}{a+x}}{a+\dfrac{ax}{a-x}}$.

3. $\dfrac{\dfrac{x}{x-1}-\dfrac{x+1}{x}}{\dfrac{x}{x+1}-\dfrac{x-1}{x}}$.

4. $\dfrac{x}{1-\dfrac{1}{1+x}}$.

5. $a+\dfrac{a}{a+\dfrac{1}{a}}$.

6. $x-\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}$.

7. $\dfrac{\dfrac{a}{a-1}+1}{1-\dfrac{a}{1-a}}$.

8. $\dfrac{x-\dfrac{1}{1+x}}{\dfrac{1-x-x^2}{x+1}}$.

9. $\dfrac{\dfrac{a^2}{b^3}-\dfrac{b^2}{a^3}}{1-\dfrac{b}{a}}$.

10. $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}$.

11. $\dfrac{\dfrac{x^2+1}{2x-1}-\dfrac{1}{2}x}{\dfrac{x+2}{1-2x}}$.

12. $\dfrac{\dfrac{a+x}{x}-\dfrac{2x}{x-a}}{\dfrac{a^2+x^2}{x-a}}$.

13. $\dfrac{\dfrac{a+x}{a}-\dfrac{x-y}{x}}{\dfrac{x^2+ay}{a^2}}$.

14. $\dfrac{1}{1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}}$.

15. $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-x}}}$.

16. $\dfrac{\dfrac{n}{n+x}-\dfrac{n}{n-x}}{\dfrac{n}{n-x}+\dfrac{n}{n+x}}$.

17. $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{4ax}{a^2-x^2}}$.

18. $\dfrac{\dfrac{a+1}{a-1}+\dfrac{a-1}{a+1}}{\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}}$.

19. $\dfrac{\dfrac{x}{x-2}-\dfrac{x}{x+2}}{\dfrac{2x}{\frac{1}{2}x^4-x^3+4x-8}}$.

20. $\dfrac{\dfrac{x^4}{x+1}-\dfrac{1}{x^4+x^5}}{x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}}$.

21. $\dfrac{\dfrac{a}{n}-\dfrac{n-x}{a}+\dfrac{ax}{n^2-nx}}{\dfrac{a}{n-x}+\dfrac{n-x}{a}+2}$.

22. $\dfrac{\left(p+\dfrac{1}{q}\right)^p\left(p-\dfrac{1}{q}\right)^q}{\left(q+\dfrac{1}{p}\right)^p\left(q-\dfrac{1}{p}\right)^q}$.

EXERCISES XII.

MISCELLANEOUS EXAMPLES.

Simplify the following expressions:

1. $$\frac{1-\left(\frac{1-a}{1+a}\right)^2}{1+\left(\frac{1-a}{1+a}\right)^2}.$$

2. $$\frac{(a-b)^2-\left(\frac{a^2+b^2}{a+b}\right)^2}{b-a+\frac{a^2}{a+b}}.$$

3. $$\frac{a+b}{ab}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{b+c}{bc}\left(\frac{1}{c}-\frac{1}{b}\right).$$

4. $$\left(\frac{a+b}{c+d}+\frac{a-b}{c-d}\right)\div\left(\frac{a+b}{c-d}+\frac{a-b}{c+d}\right).$$

5. $$a+b-\frac{1}{a+\frac{1}{b}}-\frac{1}{b+\frac{1}{a}}.$$

6. $$\frac{a}{1+\frac{a}{b}}+\frac{b}{1+\frac{b}{a}}-\frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

7. $$m-\frac{1}{1-m+m^2-\frac{m^3}{1+m}}.$$

8. $$\left(1+a-\frac{a^2+3}{a-1}\right)(1-a^2).$$

9. $$\left(\frac{x}{a+x}+a\right)\left(\frac{a}{a-x}-x\right)-\left(\frac{a}{a+x}+x\right)\left(\frac{x}{a-x}-a\right).$$

10. $$\frac{1}{1-\frac{x}{x-1}}-\frac{1}{\frac{x}{x+1}-1}.$$

11. $$\frac{a^2-x^2}{\frac{1}{a^2}-\frac{2}{ax}+\frac{1}{x^2}}\times\frac{\frac{1}{a^2x^2}}{a+x}.$$

12. $$\left(\frac{n-1}{n+1}-\frac{n+1}{n-1}\right)\times\left(\frac{1}{2}-\frac{n}{4}-\frac{1}{4n}\right).$$

13. $$\frac{\frac{a^2}{a+n}-\frac{a^3}{a^2+n^2+2an}}{\frac{a}{a+n}-\frac{a^2}{a^2-n^2}}.$$

14. $$\frac{\frac{ab+1}{b}}{a+\frac{1}{\frac{bc+1}{c}}}-\frac{1}{b(abc+a+c)}.$$

15. $$\frac{a+\frac{1}{b}}{b+\frac{1}{a}}\times\frac{b+\frac{1}{c}}{c+\frac{1}{b}}\times\frac{c+\frac{1}{a}}{a+\frac{1}{c}}.$$

16. $$\frac{\frac{1}{x}-\frac{x+a}{x^2+a^2}}{\frac{1}{a}-\frac{a+x}{a^2+x^2}}+\frac{\frac{1}{x}-\frac{x-a}{x^2+a^2}}{\frac{1}{a}-\frac{a-x}{a^2+x^2}}.$$

17. $\left[\dfrac{1}{p^2}+\dfrac{1}{q^2}+\dfrac{2}{p+q}\left(\dfrac{1}{p}+\dfrac{1}{q}\right)\right]\div(p+q)^2.$

18. $\left[\left(\dfrac{x^2}{y^3}+\dfrac{1}{x}\right)\div\left(\dfrac{x}{y^2}-\dfrac{1}{y}+\dfrac{1}{x}\right)\right]\times\dfrac{-y}{x+y}.$

19. $\left[\left(\dfrac{2x}{x^2+1}+\dfrac{2x}{x^2-1}\right)\div\left(\dfrac{x}{x^2+1}-\dfrac{x}{x^2-1}\right)\right]^2.$

20. $\left[(a^2-b^2)\div\left(\dfrac{1}{b}-\dfrac{1}{a}\right)\right]-\left[(a^2-b^2)\div\left(\dfrac{1}{b}+\dfrac{1}{a}\right)\right].$

21. $\left[\left(\dfrac{1}{a}+\dfrac{1}{b+c}\right)\div\left(\dfrac{1}{a}-\dfrac{1}{b+c}\right)\right]\times\left(1+\dfrac{b^2+c^2-a^2}{2bc}\right).$

In each of the following expressions make the indicated substitution, and simplify the result:

22. In $\left(\dfrac{m-a}{m-b}\right)^3$, let $m=\dfrac{a+b}{2}.$

23. In $1+\dfrac{b^2+c^2-a^2}{2bc}$, let $a+b+c=2s.$

24. In $\dfrac{m}{n}\left(1-\dfrac{m}{a}\right)+\dfrac{n}{m}\left(1-\dfrac{n}{a}\right)$, let $a=m+n.$

Verify each of the following identities:

25. $\dfrac{a(a-x)}{b}-\dfrac{b(b+x)}{a}=x$, when $x=a-b.$

26. $\dfrac{a(x-a)}{b+c}+\dfrac{b(x-b)}{a+c}+\dfrac{c(x-c)}{a+b}=x$, when $x=a+b+c.$

27. $(1+x)(1+y)(1+z)=(1-x)(1-y)(1-z)$, when

$$x=\frac{a-b}{a+b},\ y=\frac{b-c}{b+c},\ z=\frac{c-a}{c+a}.$$

CHAPTER VIII.

FRACTIONAL EQUATIONS IN ONE UNKNOWN NUMBER.

1. A **Fractional Equation** is an equation whose members, either or both, are fractional expressions in the unknown number or numbers.

E.g., $\dfrac{3}{x+2}=\dfrac{2}{x+1}$, $x-2+\dfrac{4-2x}{x+1}=0$.

2. Ex. 1. Solve the equation $\dfrac{3}{x+2}=\dfrac{2}{x+1}$.

Multiplying by $(x+1)(x+2)$, $3(x+1)=2(x+2)$.

Transferring terms, $3x-2x=4-3$.

Uniting terms, $x=1$.

Check: $\dfrac{3}{1+2}=\dfrac{2}{1+1}$, or $1=1$.

In clearing this equation of fractions, we multiplied by an expression, $(x+1)(x+2)$, which contains the unknown number. In such a case a root may be introduced. But if a root is introduced in clearing of fractions, it must be a root of one of the factors of the L. C. D. equated to 0. Since 1 is not a root of

$$x+1=0, \text{ or of } x+2=0,$$

it is a root of the given equation.

Ex. 2. Solve the equation $\dfrac{2x+19}{5x^2-5}-\dfrac{17}{x^2-1}=-\dfrac{3}{x-1}$.

The L.C.D. is $5(x^2-1)$, $=5(x-1)(x+1)$.

Multiplying by $5(x^2-1)$, $2x+19-85=-15x-15.$

Transferring terms, $2x+15x=-15-19+85.$

Uniting terms, $17x=51.$

Dividing by 17, $x=3.$

Since 3 is not a root of $x-1=0$, or of $x+1=0$, it is a root of the given equation.

Ex. 3. Solve the equation $\dfrac{6x+1}{4}-\dfrac{2x-1}{3x-2}=\dfrac{3x-1}{2}.$

When the denominators of some of the fractions do not contain the unknown number, it is usually better first to unite these fractions.

Transferring $\dfrac{3x-1}{2}$, $\dfrac{6x+1}{4}-\dfrac{3x-1}{2}-\dfrac{2x-1}{3x-2}=0.$

Uniting first two fractions, $\dfrac{3}{4}-\dfrac{2x-1}{3x-2}=0.$

Multiplying by $4(3x-2)$, $9x-6-8x+4=0.$

Transferring and uniting terms, $x=2.$

Since 2 is not a root of $3x-2=0$, it is a root of the given equation.

Ex. 4. If both members of the equation

$$\frac{-2x^2}{x^2-1}+\frac{x}{1-x}=-\frac{x}{x+1}-3 \qquad (1)$$

be multiplied by x^2-1, we obtain the integral equation

$$-2x^2-x(x+1)=-x(x-1)-3(x^2-1),$$

or

$$(x+1)(x-3)=0. \qquad (2)$$

Now observe that it was not necessary to multiply by x^2-1, $=(x+1)(x-1)$, to clear the given equation of fractions. For, if the terms in the second member be transferred to the first member, we have

$$\frac{-2x^2}{x^2-1}+\frac{x}{1-x}+\frac{x}{1+x}+3=0,$$

or, uniting terms, $\dfrac{x^2-2x-3}{x^2-1}=0,$

or, cancelling $x+1$, $$\frac{x-3}{x-1}=0.$$

Clearing the last equation of fractions, we have

$$x-3=0; \tag{3}$$

whence $$x=3.$$

The root 3 of the derived equation (3) is found, by substitution, to be a root of the given equation. Had we solved equation (2), we should have obtained the additional root -1, which is not a root of the given equation.

This root *was introduced by multiplying both members of the given equation by the unnecessary factor* $x+1$, and is a root of the equation obtained by equating this factor to 0.

EXERCISES I.

Solve each of the following equations:

1. $\dfrac{x+3}{x-3}=3.$ 2. $\dfrac{2x-1}{x-5}=5.$ 3. $\dfrac{x-2}{x+3}=\dfrac{3}{4}.$

4. $\dfrac{5}{x-12}=\dfrac{7}{24-x}.$ 5. $\dfrac{3}{x-8}=\dfrac{7}{x-4}.$

6. $\dfrac{7}{x+17}=-\dfrac{3}{x+7}.$ 7. $\dfrac{11}{x-7}=\dfrac{9}{2x-1}.$

8. $\dfrac{2x+3}{4}-\dfrac{x-1}{6x-8}=\dfrac{x+2}{2}.$ 9. $\dfrac{2x+1}{5}-\dfrac{3x-2}{6x+3}=\dfrac{6x-1}{15}.$

10. $\dfrac{2x+3}{7}-\dfrac{3x+5}{6x+2}=\dfrac{x+1}{14}.$ 11. $\dfrac{5x-1}{6}-\dfrac{1-2x}{1+2x}=\dfrac{2x+1}{3}.$

12. $\dfrac{6}{x+2}+\dfrac{x}{x-2}=1.$ 13. $\dfrac{5x}{x+3}-\dfrac{9}{x-2}=5.$

14. $\dfrac{x-3}{x-7}+\dfrac{x-5}{x+1}=2.$ 15. $\dfrac{x-9}{x-5}+\dfrac{x-5}{x-8}=2.$

16. $\dfrac{3}{x+2}+\dfrac{1}{x-2}=\dfrac{8}{x^2-4}.$ 17. $\dfrac{7}{x+3}+\dfrac{1}{x-3}=\dfrac{24}{x^2-9}.$

18. $\dfrac{3}{x+1}-\dfrac{x+1}{x-1}=\dfrac{x^2}{1-x^2}.$ 19. $\dfrac{5}{x+2}+\dfrac{7}{x+4}=\dfrac{12}{x+3}.$

20. $\dfrac{x}{x-3}+\dfrac{x-1}{x-5}=\dfrac{2x^2-5x-21}{x^2-8x+15}.$

21. $\dfrac{x+1}{x-3}+\dfrac{x-4}{x-6}=\dfrac{2x^2-7x-29}{x^2-9x+18}.$

22. $\dfrac{4}{x-7}+\dfrac{1}{x-9}=\dfrac{1}{x-5}+\dfrac{4}{x-8}.$

23. $\dfrac{3}{x-1}-\dfrac{1}{x+1}=\dfrac{1}{x+2}+\dfrac{1}{x-6}.$

24. $\dfrac{3x-17}{x^2-7x+12}-\dfrac{2x-11}{x^2-4x+3}=\dfrac{x-5}{x^2-5x+4}.$

25. $\dfrac{x-5}{x^2-10x+21}-\dfrac{2x-15}{x^2-12x+35}=\dfrac{7-x}{x^2-8x+15}.$

26. $\dfrac{x+\frac{1}{3}}{x-\frac{2}{3}}=\dfrac{x-1}{x-\frac{4}{3}}.$

27. $\dfrac{3x-1\frac{1}{4}}{x+\frac{1}{4}}=\dfrac{3(x+\frac{1}{4})}{x+2\frac{1}{4}}.$

28. $\dfrac{2}{3-\dfrac{2}{x+1}}=1.$

29. $\dfrac{\dfrac{5}{x-3}+\dfrac{3}{x+3}}{\dfrac{5}{x-3}-\dfrac{3}{x+3}}=\dfrac{11}{14}.$

Problems.

3. Pr. **1.** A number of men received \$120, to be divided equally. If their number had been 4 less, each one would have received three times as much. How many men were there?

Let x stand for the number of men. Then each man received $\dfrac{120}{x}$ dollars. If their number had been 4 less, each one would have received $\dfrac{120}{x-4}$ dollars.

The problem states,

in *verbal* language: *the number of dollars each would have received, if there had been four less, is equal to three times the number of dollars each received.*

in *algebraic* language: $\dfrac{120}{x-4}=3\times\dfrac{120}{x}.$

Whence, $x=6.$

Therefore there were six men.

Pr. **2**. A can do a piece of work in 9 days, B in 6 days; and A, B, and C together in 3 days. In how many days can C do the work?

Let x stand for the number of days it takes C to do the work.

Then, in one day,

A does $\frac{1}{9}$ of the work; B does $\frac{1}{6}$; and C does $\frac{1}{x}$.

In 3 days,

A does $\frac{3}{9}$ of the work; B does $\frac{3}{6}$; and C does $\frac{3}{x}$.

Therefore, in 3 days, A, B, and C together do

$$\frac{3}{9}+\frac{3}{6}+\frac{3}{x} \text{ of the work.}$$

The problem states,

in *verbal* language: *the work done by A, B, and C together in 3 days is equal to all the work, or* 1;

in *algebraic* language: $\frac{3}{9}+\frac{3}{6}+\frac{3}{x}=1.$

Whence, $x=18.$

Therefore C can do the work in 18 days.

Pr. **3**. A cistern has 3 taps. By the first it can be emptied in 80 minutes, by the second in 200 minutes, and by the third in 5 hours. After how many hours will the cistern be emptied, if all the taps are opened?

Let x stand for the number of minutes it takes the three taps together to empty the cistern.

Then, in 1 minute, the three together will empty $\frac{1}{x}$ of the cistern.

But, in 1 minute, the first will empty $\frac{1}{80}$ of the cistern; the second $\frac{1}{200}$, and the third $\frac{1}{300}$; and together they will empty $\frac{1}{80}+\frac{1}{200}+\frac{1}{300}$ of the cistern.

Therefore $$\frac{1}{80}+\frac{1}{200}+\frac{1}{300}=\frac{1}{x}.$$

Whence $x=48.$

It will take the three taps together 48 minutes, or $\frac{4}{5}$ of an hour, to empty *the cistern*.

EXERCISES II.

1. What number added to the numerator and denominator of $\frac{2}{7}$ will give a fraction equal to $\frac{3}{4}$?

2. The sum of two numbers is 18, and the quotient of the less divided by the greater is equal to $\frac{1}{5}$. What are the numbers?

3. The denominator of a fraction exceeds its numerator by 2, and if 1 be added to both numerator and denominator, the resulting fraction will be equal to $\frac{2}{3}$. What is the fraction?

4. The sum of a number and 7 times its reciprocal is 8. What is the number?

5. The value of a fraction, when reduced to its lowest terms, is $\frac{3}{7}$. If its numerator be increased by 7 and its denominator be decreased by 7, the resulting fraction will be equal to $\frac{2}{3}$. What is the fraction?

6. What number must be added to the numerator and subtracted from the denominator of the fraction $\frac{7}{13}$, to give its reciprocal?

7. If $\frac{1}{4}$ be divided by a certain number increased by $\frac{1}{4}$, and $\frac{1}{4}$ be subtracted from the quotient, the remainder will be $\frac{1}{4}$. What is the number?

8. A train runs 200 miles in a certain time. If it were to run 5 miles an hour faster, it would run 40 miles farther in the same time. What is the rate of the train?

9. A number has three digits, which increase by 1 from left to right. The quotient of the number divided by the sum of the digits is 26. What is the number?

10. A number of men have \$72 to divide. If \$144 were divided among 3 more men, each one would receive \$4 more. How many men are there?

11. It was intended to divide $\frac{1}{2}$ by a certain number, but by mistake $\frac{1}{2}$ was added to the number. The result was, nevertheless, the same. What is the number?

12. A steamer can run 20 miles an hour in still water. If it can run 72 miles with the current in the same time that it can run 48 miles against the current, what is the speed of the current?

13. A man buys two kinds of wine, 14 bottles in all, paying $9 for one kind and $12 for the other. If the price of each kind is the same, how many bottles of each does he buy?

14. A farmer intended to feed 80 bushels of corn to a certain number of sheep. When 6 of the sheep died, he could have sold 24 bushels of corn and have had enough left to give each remaining sheep the same amount as before. How many sheep had he?

15. It takes a pedestrian 5 hours to go from A to B. It takes a bicycle rider, who goes 6 miles farther every hour, 2 hours to go the same distance. How far is A from B?

16. A can do a piece of work in 10 days, B in 6 days and A, B, and C together in 3 days. In how many days can C do the work?

17. A and B together can do a piece of work in 2 days, B and C together in 3 days, and A and C together in $2\frac{1}{2}$ days. In how many days can A, B, and C together do the work?

18. The circumference of the hind wheel of a carriage exceeds the circumference of the front wheel by 4 feet, and the front wheel makes the same number of revolutions in running 400 yards that the hind wheel makes in running 500 yards. What is the circumference of each wheel?

19. A cistern has 3 taps. By the first it can be filled in 6 hours, by the second in 8 hours, and by the third it can be emptied in 12 hours. In what time will it be filled if all the taps are opened?

20. An inlet pipe can fill a cistern in 3 hours, and an outlet pipe can empty it in 9 hours. After how many hours will the cistern be filled if both pipes are open one-half of the time, and the outlet pipe is closed during the second half of the time?

21. In a number of two digits, the digit in the tens' place exceeds the digit in the units' place by 2. If the digits be interchanged and the resulting number be divided by the original number, the quotient will be equal to $\frac{23}{32}$. What is the number?

22. In a number of three digits, the digit in the hundreds' place is 2; if this digit be transferred to the units' place, and the resulting number be divided by the original number, the quotient will be equal to $\frac{77}{41}$. What is the number?

23. In one hour a train runs 10 miles farther than a man rides on a bicycle in the same time. If it takes the train 6 hours longer to run 255 miles than it takes the man to ride 63 miles, what is the rate of the train?

24. A cistern has three pipes. To fill it, the first pipe takes one-half of the time required by the second, and the second takes two-thirds of the time required by the third. If the three pipes be open together, the cistern will be filled in 6 hours. In what time will each pipe fill the cistern?

25. A and B ride 100 miles from P to Q. They ride together at a uniform rate until they are within 30 miles of Q, when A increases his rate by $\frac{1}{5}$ of his previous rate. When B is within 20 miles of Q, he increases his rate by $\frac{1}{2}$ of his previous rate, and arrives at Q 10 minutes earlier than A. At what rate did A and B first ride?

26. A circular road has three stations, A, B, and C, so placed that A is 15 miles from B, B is 13 miles from C in the same direction, and C is 14 miles from A in the same direction. Two messengers leaving A at the same time, and travelling in opposite directions, meet at B. The faster messenger then reaches A 7 hours before the slower one. What is the rate of each messenger?

CHAPTER IX.

LITERAL EQUATIONS IN ONE UNKNOWN NUMBER.

1. The unknown numbers of an equation are frequently to be determined in terms of general numbers, *i.e.*, in terms of numbers represented by letters. The latter are commonly represented by the leading letters of the alphabet, a, b, c, etc.

Such numbers as a, b, c, etc., are to be regarded as known.

E.g., in the equation $x + a = b$, a and b are the *known* numbers, and x is the *unknown* number.

From this equation we obtain $x = b - a$.

2. A **Numerical Equation** is one in which all the known numbers are numerals; as $2x + 3 = 7$; $4x - 3y = 7$.

A **Literal Equation** is one in which some or all of the known numbers are literal; as $2ax + 3b = 5$; $ax + by = c$.

3. **Ex. 1.** Solve the equation $\dfrac{x-a}{b} + \dfrac{x-b}{a} = -\dfrac{(a-b)^2}{2ab}$.

Clearing of fractions,

$$2ax - 2a^2 + 2bx - 2b^2 = -a^2 + 2ab - b^2.$$

Transferring and uniting terms,

$$2(a+b)x = a^2 + 2ab + b^2.$$

Dividing by $2(a+b)$, $\quad x = \dfrac{a+b}{2}$.

Notice that the above equation, although algebraically fractional, is integral in the unknown number x. The equation which follows is fractional in the unknown number.

Ex. 2. Solve the equation $\qquad \dfrac{a+x}{b+x}=\dfrac{a+1}{b+1}.$

Multiplying by $(b+x)(b+1)$, $(a+x)(b+1)=(b+x)(a+1)$.

Simplifying, $\qquad ab+bx+a+x=ab+ax+b+x.$

Cancelling terms, $\qquad bx+a=ax+b.$

Transferring and uniting terms, $\qquad (b-a)x=b-a.$

Dividing by $b-a$, $\qquad x=1.$

EXERCISES I.

Solve the following equations:

1. $a-x=c.$ 2. $mx+a=b.$ 3. $mx=nx+2.$

4. $3ax-5ab+6ax-7ac=2ax+2ab.$

5. $4a^2-2abx+b^2+3a^2x=5a^2-b^2x+2a^2x.$

6. $a(x+a)-b(x-b)=3ax+(a-b)^2.$

7. $x(x+a)+x(x+b)-2(x+a)(x+b)=0.$

8. $a+\dfrac{b}{x}=c.$ 9. $\dfrac{a}{b}=\dfrac{x-b^2}{x-a^2}.$

10. $\dfrac{x+a}{x-a}=\dfrac{5}{4}.$ 11. $\dfrac{b^2}{ax}+\dfrac{b}{a}-\dfrac{a}{b}=\dfrac{a}{x}.$

12. $\dfrac{a+x}{b+x}=\dfrac{a+1}{b+1}.$ 13. $\dfrac{x+a}{2}-\dfrac{2}{x+a}=\dfrac{x-a}{2}.$

14. $\dfrac{6x+a}{4x+b}-\dfrac{3x-b}{2x-a}=0.$ 15. $\dfrac{a+x}{b+a}=\dfrac{a-x}{b-a}.$

16. $\dfrac{x+ab}{x-ab}=\dfrac{a^2+ab+b^2}{a^2-ab+b^2}.$ 17. $\dfrac{x+a}{x-b}=\dfrac{(2x+a)^2}{(2x-b)^2}.$

18. $\dfrac{x^2+a^2}{4x^2-a^2}-\dfrac{x}{2x+a}=-\dfrac{1}{4}.$

19. $\dfrac{a(x+1)-b(x-1)}{b(x+1)-a(x-1)}=\dfrac{a^3}{b^3}.$

20. $\dfrac{a^3-b^3}{a^3+b^3}=\dfrac{a(x-b^2)+b(a^2-x)}{a(x-b^2)-b(a^2-x)}.$

21. $\dfrac{x-a}{2bc}+\dfrac{x-b}{2ac}+\dfrac{x-c}{2ab}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}.$

22. $\dfrac{1-2ax^2}{1+2bx^2}-\dfrac{1+2ax^2}{1-2bx^2}=\dfrac{4abx^3}{4b^2x^4-1}.$

23. $\dfrac{a^2+4a}{x^2+x-a^2+a}-\dfrac{a}{x+a}=\dfrac{1}{x-a+1}.$

24. $\dfrac{a^2+x}{b^2-x}-\dfrac{a^2-x}{b^2+x}=\dfrac{4abx+2a^2-2b^2}{b^4-x^2}.$

25. $\dfrac{a^2+ax+x^2}{a^3+a^2x+ax^2+x^3}-\dfrac{a^3-a^2x+ax^2}{a^4+2a^2x^2+x^4}=\dfrac{1}{a+x}.$

26. $\dfrac{a^2-x}{x-2a}-\dfrac{2a+x}{a^2-x}=\dfrac{a^4}{a^2x+2ax-2a^3-x^2}.$

27. $\dfrac{a+\dfrac{x}{a-b}}{a-\dfrac{x}{a+b}}-1=\dfrac{2a}{b}.$ 28. $\dfrac{a+1}{a+\dfrac{a+b}{a+\dfrac{b^2}{x-a}}}=1.$

General Problems.

4. A **General Problem** is one in which the *known* numbers are *literal.*

Pr. 1. The greater of two numbers is m times the less, and their sum is s. What are the numbers?

Let x stand for the less required number. Then mx stands for the greater. By the condition of the problem, we have

$$x+mx=s;$$

whence, $x=\dfrac{s}{1+m}$, the less number, and $mx=\dfrac{ms}{1+m}$, the greater.

If $m=3$ and $s=84$, we have

$$x=\frac{84}{1+3}=21, \text{ and } mx=3\times 21=63.$$

When the numbers are equal, $m=1$, and we obtain

$$x=\frac{s}{2}, \text{ and } mx=\frac{s}{2},$$

for all values of s; that is, either of the two numbers is half their sum.

Thus the solution of this general problem includes the solutions of all like problems. A solution for any like problem is obtained by substituting particular values for m and s, as above.

Pr. 2. A cistern has two taps. By the first it can be filled in a minutes, and by the second in b minutes. How many minutes will it take the two taps together to fill the cistern?

Let x stand for the number of minutes it takes the two taps to fill the cistern. Then, in 1 minute, the two together will fill $\frac{1}{x}$ of the cistern.

But, in 1 minute, the first will fill $\frac{1}{a}$ of the cistern, the second $\frac{1}{b}$; and together they will fill $\frac{1}{a}+\frac{1}{b}$ of the cistern.

Therefore $$\frac{1}{a}+\frac{1}{b}=\frac{1}{x}.$$

Whence $$x=\frac{ab}{a+b}.$$

This solution gives a general rule for solving problems of like character. In a particular example, a may be the number of minutes it takes a tap to fill a cistern, the number of hours it takes a man to build a wall, to dig a ditch, to plough a field, etc.

Pr. 3. If one man can dig a ditch in 6 days, and a second man in 3 days, in how many days can they dig the ditch, working together?

Substituting $a=6$, $b=3$, in the result of Pr. 2, we have

$$x=\frac{6\times 3}{6+3}=2.$$

Therefore they can together dig the ditch in 2 days.

EXERCISES II.

Find the general solution of each of the following problems, and from this solution obtain the particular solution for the numerical values assigned to the literal numbers in the problem.

1. Find a number, such that the result of adding it to n shall be equal to n times the number. Let $n = 2$; 5.

2. Divide a into two parts, such that $\frac{1}{m}$ of the first, plus $\frac{1}{n}$ of the second, shall be equal to b. Let $a = 100$, $b = 30$, $m = 3$, $n = 5$.

3. A sum of d dollars is divided between A and B. B receives b dollars as often as A receives a dollars. How much does each receive? Let $d = 7000$, $a = 3$, $b = 2$.

4. A father's age exceeds his son's age by m years, and the sum of their ages is n times the son's age. What are their ages? Let $m = 20$, $n = 4$; $m = 25$, $n = 7$.

5. A farmer can plough a field in a days, and his son in b days; in how many days can they plough the field, working together? Let $a = 10$, $b = 15$.

6. What time is it, if the number of hours which have elapsed since noon is m times the number of hours to midnight? Let $m = \frac{1}{2}$.

7. One pipe can fill a cistern in a hours, a second in b hours, and a third in c hours. In how many hours can the three pipes fill the cistern, working together? Let $a = 2$, $b = 3$, $c = 6$.

8. One pipe can fill a cistern in m hours, a second in n hours, and a third can empty it in p hours. After how many hours will the cistern be filled, if all pipes are open? Let $m = 4$, $n = 6$, $p = 3$.

9. Two couriers start at the same time and move in the same direction, the first from a place d miles ahead of the second. The first courier travels at the rate of m_1 miles an hour, and the second at the rate of m_2 miles an hour. After

how many hours will the second courier overtake the first? Let $d = 15$, $m_1 = 17$, $m_2 = 20$.

From the result of the preceding example find the results of Exx. 10–12.

10. At what rate must the second courier travel in order to overtake the first after h hours? Let $d = 18$, $m_1 = 15$, $h = 3$.

11. At what rate must the first courier travel in order that the second courier may overtake him after h hours? Let $d = 12$, $m_2 = 22$, $h = 3$.

12. How many miles behind the first courier must the second start in order to overtake the first after h hours? Let $m_1 = 18$, $m_2 = 21$, $h = 4$.

13. In a company are a men and b women; and to every m unmarried men there are n unmarried women. How many married couples are in the company? Let $a = 13$, $b = 17$, $m = 3$, $n = 5$.

INTERPRETATION OF THE SOLUTIONS OF PROBLEMS.

5. In solving equations we do not concern ourselves with the meaning of the results. When, however, an equation has arisen in connection with a problem, the interpretation of the result becomes important. In this chapter we shall interpret the solutions of some linear equations in connection with the problems from which they arise.

Positive Solutions.

6. Pr. A company of 20 people, men and women, proposed to arrange a fair for the benefit of a poor family. Each man contributed \$ 3, and each woman \$ 1. If \$ 55 were contributed, how many men and how many women were in the company?

Let x stand for the number of men; then the number of women was $20 - x$. The amount contributed by the men was $3x$ dollars, that by the women $20 - x$ dollars. By the condition of the problem, we have

$$3x + (20 - x) = 55; \text{ whence } x = 17\tfrac{1}{2}.$$

The result, $17\frac{1}{2}$, satisfies the equation, but not the problem. For the number of men must be an *integer*. This implied condition could not be introduced into the equation.

The conditions stated in the problem are impossible, since they are inconsistent with the implied condition.

Negative Solutions.

7. Pr. A father is 40 years old, and his son 10 years old. After how many years will the father be seven times as old as his son?

Let x stand for the required number of years. Then after x years the father will be $40 + x$ years old, and the son $10 + x$ years old. By the condition of the problem, we have

$$40 + x = 7(10 + x), \text{ whence } x = -5. \qquad (1)$$

This result satisfies the equation, but not the condition of the problem. For since the question of the problem is "*after* how many years?" the result, if added to the number of years in the ages of father and son, should increase them, and therefore be *positive*. Consequently, at no time in the future will the father be seven times as old as his son. But since to add -5 is equivalent to subtracting 5, we conclude that the question of the problem should have been, "How many years ago?"

The equation of the problem, with this modified question, is:

$$40 - x = 7(10 - x); \text{ whence } x = 5. \qquad (2)$$

Notice that equation (2) could have been obtained from equation (1) by changing x into $-x$.

8. The interpretation of a negative result in a given problem is often facilitated by the following principle:

If $-x$ be substituted for x in an equation which has a negative root, the resulting equation will have a positive root of the same absolute value; and vice versa.

E.g., the equation $x + 1 = -x - 3$ has the root -2; while the equation $-x + 1 = x - 3$ has the root 2.

9. Pr. Two pocket-books contain together $100. If one-half of the contents of one pocket-book and one-third of the contents of the other be removed, the amount of money left in both will be $70. How many dollars does each pocket-book contain?

Let x stand for the number of dollars contained in the first pocket-book; then the number of dollars contained in the second is $100 - x$. When one-half of the contents of the first and one-third of the contents of the second are removed, the number of dollars remaining in the first is $\frac{1}{2}x$, and in the second $\frac{2}{3}(100 - x)$. By the conditions of the problem, we have

$$\tfrac{1}{2}x + \tfrac{2}{3}(100 - x) = 70, \text{ whence } x = -20.$$

Substituting $-x$ for x in the given equation, we obtain

$$-\tfrac{1}{2}x + \tfrac{2}{3}(100 + x) = 70, \text{ or } \tfrac{2}{3}(100 + x) - \tfrac{1}{2}x = 70.$$

This equation corresponds to the following conditions:

If x stand for the number of dollars in one pocket-book, then $100 + x$ stands for the number of dollars in the other; that is, one pocket-book contains $100 more than the other. The second condition of the problem, obtained from the equation, is: two-thirds of the contents of one pocket-book exceeds one-half of the contents of the other by $70. Therefore the modified problem reads as follows:

Two pocket-books contain a certain amount of money, and one contains $100 more than the other. If one-third of the contents be removed from the first pocket-book, and one-half of the contents from the second, the first will then contain $70 more than the second. How much money is contained in each pocket-book?

10. These problems show that the required modification of an assumption, question, or condition of a problem which has led to a negative result, consists in making the assumption, question, or condition the opposite of what it originally was.

Thus, if a positive result signify a distance toward the right from a certain point, a negative result will signify a distance *toward the left* from the same point; and *vice versa;* etc.

Zero Solutions.

11. A zero result gives in some cases the answer to the question; in other cases it proves its impossibility.

Pr. A merchant has two kinds of wine, one worth $7.25 a gallon, and the other $5.50 a gallon. How many gallons of each kind must be taken to make a mixture of 16 gallons worth $88?

Let x stand for the number of gallons of the first kind; then $16 - x$ will stand for the number of gallons of the second kind.

Therefore, by the condition of the problem, we have

$$7.25x + 5.5(16 - x) = 88; \text{ whence } x = 0.$$

That is, no mixture which contains the first kind of wine can be made to satisfy the condition. In fact, 16 gallons of the second kind are worth $88.

EXERCISE III.

Solve the following problems, and interpret the results. Modify those problems which have negative solutions so that they will be satisfied by positive solutions.

1. A and B together have $100. If A spend one-third of his share, and B spend one-fourth of his share, they will then have $80 left. What are their respective shares?

2. A father is 40 years old, and his son is 13 years old; after how many years will the father be four times as old as his son?

3. The sum of the first and third of three consecutive numbers is equal to three times the second. What are the numbers?

4. In a number of two digits, the tens' digit is two-thirds of units' digit. If the digits be interchanged, the resulting number will exceed the original number by 36. What is the number?

5. A teacher proposes 30 problems to a pupil. The latter is to receive 8 marks in his favor for each problem solved, and 12 marks against him for each problem not solved. If the number of marks against him exceed those in his favor by 420, how many problems *will he have* solved?

6. In a number of two digits the tens' digit is twice the units' digit. If the digits be interchanged, the resulting number will exceed the original number by 18. What is the number?

7. A has \$100, and B has \$30. A spends twice as much money as B, and then has left three times as much as B. How much does each one spend?

Discuss the solutions of the following general problems. State under what conditions each solution is positive, negative, or zero. Also, in each problem, assign a set of particular values to the general numbers which will give an admissible solution.

8. A father is a years old, and his son is b years old. After how many years will the father be n times as old as his son?

9. Having two kinds of wine worth a and b dollars a gallon, respectively, how many gallons of each kind must be taken to make a mixture of n gallons worth c dollars a gallon?

10. Two couriers, A and B, start at the same time from two stations, distant d miles from each other, and travel in the same direction. A travels n times as fast as B. Where will A overtake B?

CHAPTER X.

SIMULTANEOUS LINEAR EQUATIONS.

SYSTEMS OF EQUATIONS.

1. If the linear equation in two unknown numbers

$$x + y = 5 \qquad (1)$$

be solved for y, we obtain

$$y = 5 - x.$$

We may substitute in this equation any particular numerical value for x, and obtain a corresponding value for y. Thus,

when $x = 1$, $y = 4$; when $x = 2$, $y = 3$; when $x = 3$, $y = 2$; etc.

In like manner the equation could have been solved for x in terms of y, and corresponding sets of values obtained.

Any set of corresponding values of x and y satisfies the given equation, and is therefore a solution.

2. Solving the equation

$$y - x = 1 \qquad (2)$$

for y, we have $y = 1 + x$. Then,

when $x = 1$, $y = 2$; when $x = 2$, $y = 3$; when $x = 3$, $y = 4$; etc.

Now, observe that equations (1) and (2) have the common solution, $x = 2$, $y = 3$. It seems evident that no other set of values of x and y will satisfy both of these equations, which therefore have only this solution in common.

Equations (1) and (2) express different relations between the unknown numbers, and are called **Independent Equations.**

Also since they are satisfied by a common set of values of the unknown numbers, they are called **Consistent Equations.**

3. A **System of Simultaneous Equations** is a group of equations which are to be satisfied by the same set, or sets, of values of the unknown numbers.

A **Solution** of a system of simultaneous equations is a set of values of the unknown numbers which satisfies all of the equations.

4. The examples of Arts. 1–2 are illustrations of the following general principles:

A system of linear equations has a definite number of solutions.

(i.) *When the number of equations is the same as the number of unknown numbers.*

(ii.) *When the equations are independent and consistent.*

5. *Two systems of equations are equivalent when every solution of either system is a solution of the other.*

E.g., the systems (I.) and (II.):

$$\left.\begin{aligned} 3x + 2y &= 8, \\ x - y &= 1, \end{aligned}\right\} \text{(I.)} \qquad \left.\begin{aligned} 3x + 2y &= 8, \\ 2x - 2y &= 2, \end{aligned}\right\} \text{(II.)}$$

are equivalent. For they are both satisfied by the solution, $x = 2$, $y = 1$, and, as we shall see later, by no other solution.

6. If the equations $\quad x + y = 7,$

$$x - y = 1,$$

be added, we obtain $\quad 2x = 8,$

in which the unknown number y does not appear. We say that y has been *eliminated* from the given equations.

7. Elimination is the process of deriving from two or more equations an equation which has one less unknown number.

Elimination by Addition and Subtraction.

8. Ex. 1. Solve the system $3x + 4y = 24$, (1)

$$5x - 6y = 2. \qquad (2)$$

To eliminate y, we multiply the equations by such numbers *as will make the* coefficients of y numerically equal.

Multiplying (1) by 3, $9x + 12y = 72.$ (3)

Multiplying (2) by 2, $10x - 12y = 4.$ (4)

Adding (3) and (4), $19x = 76.$ (5)

Whence $x = 4.$

Substituting 4 for x in (1), $12 + 4y = 24.$

Whence $y = 3.$

The equations (3)–(5) are equivalent to the given equations (1)–(2).

Consequently the required solution is $x = 4$, $y = 3$.

This solution may be written 4, 3, it being understood that the first number is the value of x, and the second the value of y.

Ex. 2. Solve the system $12x + 15y = 8.$ (1)

$$16x + 9y = 7. \quad (2)$$

We will first eliminate x.

Multiplying (1) by 4, $48x + 60y = 32.$ (3)

Multiplying (2) by 3, $48x + 27y = 21.$ (4)

Subtracting (4) from (3), $33y = 11.$

Whence $y = \frac{1}{3}.$

Substituting $\frac{1}{3}$ for y in (1), $12x + 5 = 8.$

Whence $x = \frac{1}{4}.$

Consequently the required solution is $\frac{1}{4}$, $\frac{1}{3}$.

9. The examples of the preceding article illustrate the following method of elimination by addition and subtraction:

Multiply both members of the equations by such numbers as will make the coefficients of one of the unknown numbers numerically equal. Subtract, or add, corresponding members of the resulting equations, and equate the results.

Solve this equation in one unknown number. Substitute the value of this unknown number in the simpler of the given equations, and solve for the other unknown number.

The multipliers are obtained by dividing the L. C. M. of the coefficients of the unknown number to be eliminated by the coefficients of this unknown number. It is better to eliminate that unknown number which requires the smallest multipliers.

EXERCISES I.

Solve the following systems of equations by the method of addition and subtraction:

1. $\begin{cases} x+y=17, \\ x-y=7. \end{cases}$
2. $\begin{cases} x+y=a, \\ x-y=b. \end{cases}$
3. $\begin{cases} x-12y=3, \\ x+4y=19. \end{cases}$
4. $\begin{cases} 3x+y=31, \\ 5x-2y=15. \end{cases}$
5. $\begin{cases} 4x-7y=19, \\ x+9y=37. \end{cases}$
6. $\begin{cases} 10x-3y=25, \\ 5x-9y=-25. \end{cases}$
7. $\begin{cases} nx-ay=0, \\ n^2x-ay=an. \end{cases}$
8. $\begin{cases} 5x+4y=49\frac{1}{2}, \\ 2x+7y=63. \end{cases}$
9. $\begin{cases} 5x-3y=12, \\ 19x-5y=73\frac{1}{3}. \end{cases}$
10. $\begin{cases} 12x+15y=8, \\ 16x+9y=7. \end{cases}$
11. $\begin{cases} ax+by=c, \\ mx+ny=p. \end{cases}$
12. $\begin{cases} 3x+16y=5, \\ -5x+28y=19. \end{cases}$
13. $\begin{cases} 21x+8y=-66, \\ 28x-23y=13. \end{cases}$
14. $\begin{cases} 18x-20y=1, \\ 15x+16y=9. \end{cases}$
15. $\begin{cases} 12x-14y=-4, \\ 8x-21y=-8.5. \end{cases}$
16. $\begin{cases} 15x-14y=33, \\ 20x+21y=-24. \end{cases}$
17. $\begin{cases} 25x+24y=98, \\ 15x-16y=-2. \end{cases}$
18. $\begin{cases} 40x-63y=57, \\ 35x-18y=87. \end{cases}$
19. $\begin{cases} 15x+28y=58a, \\ 18x-35y=a. \end{cases}$

Elimination by Comparison.

10. Ex. Solve the system

$$7x+2y=20, \tag{1}$$

$$13x-3y=17. \tag{2}$$

To eliminate y, we proceed as follows:

Solving (1) for y, $$y=\frac{20-7x}{2}. \tag{3}$$

Solving (2) for y, $$=\frac{13x-17}{3}. \tag{4}$$

Equating these values of y,

$$\frac{20-7x}{2}=\frac{13x-17}{3}. \tag{5}$$

Whence $x = 2.$

Substituting 2 for x in (3), $y = \frac{20 - 14}{2} = 3.$

The equations (3)–(5) are equivalent to the given equations (1)–(2).

Consequently the required solution is 2, 3.

11. This example illustrates the following method of elimination by comparison:

Solve the given equations for the unknown number to be eliminated, and equate the expressions thus obtained. The derived equation will contain but one unknown number.

Solve this derived equation; and substitute the value of the unknown number thus obtained in the simplest of the preceding equations, and solve for the other unknown number.

EXERCISES II.

Solve the following systems of equations by the method of comparison:

1. $\begin{cases} x = 3y - 2, \\ x = 5y - 12. \end{cases}$

2. $\begin{cases} 5y = 2x + 1, \\ 8y = 5x - 11. \end{cases}$

3. $\begin{cases} 5x + 9y = 28, \\ 7x + 3y = 20. \end{cases}$

4. $\begin{cases} 21x - 23y = 2, \\ 7x - 19y = 12. \end{cases}$

5. $\begin{cases} \frac{1}{5}x = \frac{1}{3}y - 1, \\ \frac{1}{5}y = \frac{1}{4}x - 2. \end{cases}$

6. $\begin{cases} 2\frac{1}{2}x - 3\frac{1}{8}y = 10, \\ 7\frac{1}{3}x - 5\frac{1}{2}y = 55. \end{cases}$

7. $\begin{cases} \frac{1}{7}x + 7y = 99, \\ \frac{1}{7}y + 7x = 51. \end{cases}$

8. $\begin{cases} \frac{1}{2}x + \frac{1}{6}y = 11, \\ \frac{1}{5}x + \frac{1}{24}y = \frac{7}{2}. \end{cases}$

9. $\begin{cases} 4x - 3y = 1, \\ 3x - 4y = 6. \end{cases}$

10. $\begin{cases} 8x + 3y = 58, \\ 3x - 8y = -33. \end{cases}$

11. $\begin{cases} 5x + 3y = 21, \\ 6x - 7y = 4. \end{cases}$

12. $\begin{cases} 2x - y = 5, \\ 5x - 2y = 14. \end{cases}$

13. $\begin{cases} 7x - 5y = 3, \\ 8x + 9y = -26. \end{cases}$

14. $\begin{cases} 8x + 9y = 26, \\ 32x - 3y = 26. \end{cases}$

15. $\begin{cases} 63x - 46y = 29, \\ 42x - 69y = 96. \end{cases}$ 16. $\begin{cases} x + ay + 1 = 0, \\ y + c(x+1) = 0. \end{cases}$

17. $\begin{cases} 5x + 4y = 9a - b, \\ 7x - 6y = a - 13b. \end{cases}$

18. $\begin{cases} ax - by = a^2 + b^2, \\ (a-b)x + (a+b)y = 2(a^2 - b^2). \end{cases}$

Elimination by Substitution.

12. Ex. Solve the system

$$5x - 2y = 1, \quad (1)$$

$$4x + 5y = 47. \quad (2)$$

If we wish to eliminate x, we proceed as follows:

Solving (1) for x, $x = \dfrac{1+2y}{5}$. (3)

Substituting $\dfrac{1+2y}{5}$ for x in (2),

$$4\left(\frac{1+2y}{5}\right) + 5y = 47. \quad (4)$$

Whence $y = 7$.

Substituting 7 for y in (3), $x = 3$.

It seems evident that equations (3)–(4) are equivalent to the given equations (1)–(2).

Consequently the required solution is 3, 7.

13. This example illustrates the following method of elimination by substitution:

Solve the simpler equation for the unknown number to be eliminated in terms of the other. Substitute the value thus obtained in the other equation. The derived equation will contain but one unknown number.

Solve the derived equation, and substitute the value of the unknown number thus obtained in the expression for the other unknown number, and solve for the other unknown number.

EXERCISES III.

Solve the following systems of equations by the method of substitution:

1. $\begin{cases} 5x - 2y = 21, \\ y = x. \end{cases}$ 2. $\begin{cases} ax + by = c, \\ x = y. \end{cases}$ 3. $\begin{cases} 3x + 5y = 26, \\ 2x = y. \end{cases}$

4. $\begin{cases} x = 2y - 3, \\ y = 2x - 15. \end{cases}$ 5. $\begin{cases} x = 3y - 7, \\ y = 3x - 19. \end{cases}$ 6. $\begin{cases} \frac{1}{2}y - 2x = 5, \\ y = 14x. \end{cases}$

7. $\begin{cases} 3x + 2y = 44, \\ 5x = 4y. \end{cases}$ 8. $\begin{cases} x + y = m, \\ x - ny = 0. \end{cases}$

9. $\begin{cases} 7x - 4y = 12, \\ 8x - 5y = 0. \end{cases}$ 10. $\begin{cases} 5x + 7y = 19, \\ -x + 2y = 3. \end{cases}$

11. $\begin{cases} 4x - 5y = 12, \\ 3x - y = -2. \end{cases}$ 12. $\begin{cases} 5x = 8y - 11, \\ 6y = 7x - 21. \end{cases}$

13. $\begin{cases} 7x - 3 = 5y, \\ 7y - 3 = 8x. \end{cases}$ 14. $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 2, \\ 2x + 3y = 60. \end{cases}$

15. $\begin{cases} ay = bx, \\ a + y = b + x. \end{cases}$ 16. $\begin{cases} 3x + 4y = 2, \\ 9x + 20y = 8. \end{cases}$

17. $\begin{cases} 4x - 15y = 22, \\ 6x + 7y = -26. \end{cases}$ 18. $\begin{cases} 10x - 21y = 75, \\ 15x - 14y = 35. \end{cases}$

Linear Equations in Three Unknown Numbers.

14. The following examples will illustrate a method of solving systems of three linear equations in three unknown numbers:

Ex. 1. Solve the system

$$2x - 3y + 5z = 11, \quad (1)$$
$$5x + 4y - 6z = -5, \quad (2)$$
$$-4x + 7y - 8z = -14. \quad (3)$$

To eliminate x, we proceed as follows:

Multiplying (1) by 5, $\quad 10x - 15y + 25z = 55. \quad (4)$

Multiplying (2) by 2, $\quad 10x + 8y - 12z = -10. \quad (5)$

Subtracting (4) from (5), $23y - 37z = -65.$ (6)

Multiplying (1) by 2, $4x - 6y + 10z = 22.$ (7)

Adding (3) and (7), $y + 2z = 8.$ (8)

Solving (6) and (8), $y = 2.$

$z = 3.$

Substituting 2 for y and 3 for z in (1), $x = 1.$

Consequently the required solution is 1, 2, 3.

Ex. **2**. Solve the system

$$ay - cz = 0, \qquad (1)$$

$$z - x = -b, \qquad (2)$$

$$ax + by = a^2 + b(a + c). \qquad (3)$$

Notice that by eliminating z from (1) and (2) we obtain an equation in x and y, which with equation (3) gives a system of two equations in the same two unknown numbers.

Solving (2) for z, $z = x - b.$ (4)

Substituting $x - b$ for z in (1),

$$ay - cx + cb = 0. \qquad (5)$$

Multiplying (3) by a, $a^2x + aby = a^3 + a^2b + abc.$ (6)

Multiplying (5) by b, $-bcx + aby = -b^2c.$ (7)

Subtracting (7) from (6), $(a^2 + bc)x = a^3 + a^2b + abc + b^2c$

$$= a^2(a + b) + bc(a + b)$$

$$= (a^2 + bc)(a + b); \qquad (8)$$

whence $x = a + b.$

Substituting $a + b$ for x in (4), $z = a.$

Substituting a for z in (1), $y = c.$

15. These examples illustrate the following method:

Eliminate one of the unknown numbers from any two of the equations; next eliminate the same unknown number from the third equation and either of the other two. Two equations in the same two unknown numbers are thus derived.

Solve these equations for the two unknown numbers, and substitute the values thus obtained in the simplest equation which contains the third unknown number.

EXERCISES IV.

Solve the following systems of equations:

1. $\begin{cases} x+3y+3z=19, \\ 4y=3x, \\ x=2z. \end{cases}$

2. $\begin{cases} x+y+z=21, \\ 6x=5z, \\ 3y+2z=0. \end{cases}$

3. $\begin{cases} 3x-4y+5z=18, \\ x=4z-17, \\ y=5z-21. \end{cases}$

4. $\begin{cases} 5y+4z-7x=-44, \\ z=5y-33, \\ x=5y-22. \end{cases}$

5. $\begin{cases} x+y=28, \\ x+z=30, \\ y+z=32. \end{cases}$

6. $\begin{cases} x+y=2c, \\ x+z=2b, \\ y+z=2a. \end{cases}$

7. $\begin{cases} x-y=2, \\ y-z=3, \\ x+z=9. \end{cases}$

8. $\begin{cases} 3x-y=7, \\ 3y-z=5, \\ 3z-x=0. \end{cases}$

9. $\begin{cases} 3x+5y=35, \\ 3y+5z=27, \\ 3z+5x=34. \end{cases}$

10. $\begin{cases} 3x+2y-4z=15, \\ 5x-3y+2z=28, \\ 3y+4z-x=24. \end{cases}$

11. $\begin{cases} x+y-z=c, \\ x+z-y=b, \\ y+z-x=a. \end{cases}$

12. $\begin{cases} 4x-3y+2z=9, \\ 2x+5y-3z=4, \\ 5x+6y-2z=18. \end{cases}$

13. $\begin{cases} 2x-4y+9z=28, \\ 7x+3y-5z=3, \\ 9x+10y-11z=4. \end{cases}$

14. $\begin{cases} x-2y+3z=6, \\ 2x+3y-4z=20, \\ 3x-2y+5z=26. \end{cases}$

15. $\begin{cases} 2x-7y+5z=3, \\ 9x+3y-20z=-45, \\ -13x+4y-30z=-95. \end{cases}$

16. $\begin{cases} 3x+25y-6z=-35, \\ 6x+10y-21z=37, \\ 8x-15y-14z=64. \end{cases}$

17. $\begin{cases} 8x-21y-9z=-61, \\ 12x-28y+15z=1, \\ 15x+49y-18z=59. \end{cases}$

18. $\begin{cases} ax + by = b^2, \\ by + cz = b^2 + c^2, \\ cz + ax = c^2. \end{cases}$

19. $\begin{cases} x + y + z = a + b + c, \\ ax = by, \\ az = cy. \end{cases}$

20. $\begin{cases} ax + by - cz = a^2 + b^2, \\ ax = abz + b^2, \\ by = abz + a^2. \end{cases}$

21. $\begin{cases} x + ay + a^2z + a^3 = 0, \\ x + by + b^2z + b^3 = 0, \\ x + cy + c^2z + c^3 = 0. \end{cases}$

16. It is frequently necessary to simplify the equations before applying one of the preceding methods:

Ex. 1. Solve the system

$$\frac{7+x}{5} - \frac{2x-y}{4} = 3y - 5, \tag{1}$$

$$\frac{4x-3}{6} + \frac{5y-7}{2} = 18 - 5x. \tag{2}$$

Clearing (1) and (2) of fractions,

$$28 + 4x - 10x + 5y = 60y - 100, \tag{3}$$

$$4x - 3 + 15y - 21 = 108 - 30x. \tag{4}$$

Transferring and uniting terms,

$$6x + 55y = 128, \tag{5}$$

$$34x + 15y = 132. \tag{6}$$

Multiplying (5) by 3, $\quad 18x + 165y = 384. \quad (7)$

Multiplying (6) by 11, $374x + 165y = 1452. \quad (8)$

Subtracting (7) from (8), $\quad 356x = 1068;$

whence, $\quad x = 3.$

Substituting (3) for x in (5), $18 + 55y = 128;$

whence, $\quad y = 2.$

Consequently, the required solution is 3, 2.

17. Certain fractional equations are to be solved for the *reciprocals* of one or both of the unknown numbers.

Ex. 2. Solve the system $\dfrac{3}{2x-3y}+\dfrac{5}{y-2}=8,$ (1)

$$\frac{7}{2x-3y}+\frac{3}{y-2}=10. \quad (2)$$

Let $2x-3y=u,\ y-2=v.$

Then (1) and (2) become $\dfrac{3}{u}+\dfrac{5}{v}=8,$ (3)

$$\frac{7}{u}+\frac{3}{v}=10. \quad (4)$$

We will solve this system for $\dfrac{1}{u}$ and $\dfrac{1}{v}$.

Multiplying (3) by 3, $\dfrac{9}{u}+\dfrac{15}{v}=24.$ (5)

Multiplying (4) by 5, $\dfrac{35}{u}+\dfrac{15}{v}=50.$ (6)

Subtracting (5) from (6), $\dfrac{26}{u}=26.$

Dividing by 26, $\dfrac{1}{u}=1,$ or $u=1.$

Substituting 1 for u in (3), $3+\dfrac{5}{v}=8,$

or $\dfrac{5}{v}=5.$

Dividing by 5, $\dfrac{1}{v}=1,$ or $v=1.$

We now have to solve the system,

$$2x-3y=1, \quad (7)$$

$$y-2=1. \quad (8)$$

From (8), $y=3.$

Substituting 3 for y in (7), $2x-9=1,$

or $2x=10.$

Dividing by 2, $x=5.$

Therefore the *required* solution is 5, 3.

Ex. 3. Solve the system,

$$\left.\begin{aligned} x+y&=xy, &(1)\\ 2x+2z&=xz, &(2)\\ 3y+3z&=yz. &(3)\end{aligned}\right\} \qquad \text{(I.)}$$

Observe that the given equations are neither linear nor fractional. Yet they can be transformed so that they will contain only the reciprocals of x, y, and z.

Dividing (1) by xy, (2) by xz, (3) by yz, we have:

$$\frac{1}{y}+\frac{1}{x}=1. \quad (4) \qquad \frac{2}{z}+\frac{2}{x}=1. \quad (5) \qquad \frac{3}{z}+\frac{3}{y}=1. \quad (6) \qquad \text{(II.)}$$

We will solve this system for $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

Multiplying (4) by 2, $\qquad \frac{2}{y}+\frac{2}{x}=2. \qquad (7)$

Subtracting (5) from (7), $\qquad \frac{2}{y}-\frac{2}{z}=1. \qquad (8)$

Solving (6) and (8) for $\frac{1}{y}$ and $\frac{1}{z}$, $\frac{1}{y}=\frac{5}{12}$, $\frac{1}{z}=-\frac{1}{12}$.

Substituting $\frac{5}{12}$ for $\frac{1}{y}$ in (4), $\qquad \frac{1}{x}=\frac{7}{12}$.

Consequently, a solution of the given system is $\frac{12}{7}$, $\frac{12}{5}$, -12.

It is important to notice that we cannot assume that the system (II.) is equivalent to the system (I.), since the equations of (II.) are derived from the equations of (I.) by dividing by expressions which contain the unknown numbers.

But if any solution of (I.) be lost by this transformation, it is a solution of the expressions (equated to 0) by which the equations of (I.) were divided; that is, of

$$xy=0, \; xz=0, \; yz=0. \qquad \text{(III.)}$$

The system (III.) has the solution 0, 0, 0, and this solution evidently satisfies the system (I.).

We therefore conclude that the given system has the two *solutions* $\frac{12}{7}$, $\frac{12}{5}$, -12, and 0, 0, 0.

EXERCISES V.

Solve the following systems of equations:

1. $\begin{cases} 3x + \dfrac{7y}{2} = 22, \\ 11y - \dfrac{2x}{5} = 20. \end{cases}$

2. $\begin{cases} \dfrac{x-1}{y-1} = \dfrac{3}{4}, \\ \dfrac{x+3}{y+3} = \dfrac{10}{13}. \end{cases}$

3. $\begin{cases} \dfrac{x-7}{3} + \dfrac{y-5}{2} = 7, \\ \dfrac{x-7}{2} + \dfrac{y-5}{3} = 8. \end{cases}$

4. $\begin{cases} \dfrac{2x+7y}{4} - \dfrac{x+7}{6} = 4, \\ \dfrac{2x+7y}{6} - \dfrac{x+7}{3} = 0. \end{cases}$

5 $\begin{cases} \dfrac{2x+1}{5} - \dfrac{3y+2}{7} = 2y - x, \\ \dfrac{3x-1}{4} + \dfrac{7y+2}{6} = 2x - y. \end{cases}$

6. $\begin{cases} \dfrac{3x-4}{2} + \dfrac{4y-1}{5} = x + y, \\ \dfrac{5x-9}{7} - \dfrac{y-2}{2} = x - y. \end{cases}$

7. $\begin{cases} \dfrac{x}{n+1} + \dfrac{y}{n-1} = \dfrac{1}{n-1}, \\ \dfrac{x}{n-1} + \dfrac{y}{n+1} = \dfrac{1}{n^2-1}. \end{cases}$

8. $\begin{cases} \dfrac{x}{m-a} + \dfrac{y}{m-b} = 1, \\ \dfrac{x}{n-a} + \dfrac{y}{n-b} = 1. \end{cases}$

9. $\begin{cases} \dfrac{5x+7y+2}{3} - \dfrac{3x+4y+7}{4} = x, \\ \dfrac{7x+3y+4}{4} - \dfrac{6x+5y+7}{5} = y. \end{cases}$

10. $\begin{cases} x - \dfrac{3x+5y}{17} + 17 = 5y + \dfrac{4x+7}{3}, \\ \dfrac{22-6y}{3} - \dfrac{5x-7}{11} = \dfrac{x+1}{6} - \dfrac{8y+5}{18}. \end{cases}$

11. $\begin{cases} \dfrac{3x+7y+1}{5} - \dfrac{2x-3y+8}{3} = 2, \\ \dfrac{5x-7y+10}{3} - \dfrac{3x+2y+6}{5} = 2. \end{cases}$

12. $\begin{cases} 6y - 6x = xy, \\ 10y + 3x = 6xy. \end{cases}$

13. $\begin{cases} 12x - 14y = 5xy, \\ 9x - 10y = 4xy. \end{cases}$

14. $\begin{cases} 7x - \dfrac{3}{y} = 16, \\ 3x - \dfrac{2}{y} = 4. \end{cases}$
15. $\begin{cases} \dfrac{3}{x} - \dfrac{4}{y} = 1, \\ \dfrac{5}{x} - \dfrac{6}{y} = 2. \end{cases}$
16. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = m, \\ \dfrac{b}{x} + \dfrac{a}{y} = n. \end{cases}$

17. $\begin{cases} \dfrac{3}{x-4} + \dfrac{4}{y-1} = 3, \\ \dfrac{9}{x-4} - \dfrac{2}{y-1} = 2. \end{cases}$
18. $\begin{cases} \dfrac{3}{x+2y} + \dfrac{x-5y}{3} = 8, \\ \dfrac{1}{4(x+2y)} - \dfrac{5y-x}{5} = 3\frac{1}{4}. \end{cases}$

19. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5, \\ \dfrac{1}{x} + \dfrac{1}{z} = 6, \\ \dfrac{1}{y} + \dfrac{1}{z} = 7. \end{cases}$
20. $\begin{cases} \dfrac{1}{z} + \dfrac{1}{y} = a, \\ \dfrac{1}{z} + \dfrac{1}{x} = b, \\ \dfrac{1}{x} + \dfrac{1}{y} = c. \end{cases}$
21. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} + \dfrac{2}{z} = 16, \\ \dfrac{1}{y} + \dfrac{2}{z} + \dfrac{2}{x} = 15, \\ \dfrac{1}{z} + \dfrac{2}{x} + \dfrac{2}{y} = 14. \end{cases}$

22. $\begin{cases} \dfrac{1}{x} + \dfrac{3}{y} + \dfrac{4}{z} = 8, \\ \dfrac{4}{x} + \dfrac{5}{y} - \dfrac{2}{z} = 16, \\ \dfrac{7}{x} - \dfrac{2}{y} + \dfrac{4}{z} = 21. \end{cases}$
23. $\begin{cases} \dfrac{3}{x} + \dfrac{4}{y} - \dfrac{8}{z} = 15, \\ \dfrac{5}{x} - \dfrac{1}{2y} + \dfrac{2}{z} = \dfrac{1}{6}, \\ \dfrac{9}{4x} + \dfrac{3}{y} + \dfrac{1}{z} = 13. \end{cases}$

EXERCISES VI.

MISCELLANEOUS EXAMPLES.

Solve the following systems of equations by the methods given in this chapter:

1. $\begin{cases} x + y = z + 10, \\ y = 2x - 13, \\ z = 2y - 11. \end{cases}$
2. $\begin{cases} yz = 2(y + z), \\ xz = 3(x + z), \\ xy = 4(x + y). \end{cases}$

3. $\begin{cases} \dfrac{x+y-1}{x-y+1} = a, \\ \dfrac{y-x+1}{x-y+1} = ab. \end{cases}$
4. $\begin{cases} \dfrac{7}{2x+3y} = \dfrac{11}{2x-3y}, \\ \dfrac{x}{10y-7} = \dfrac{9}{10}. \end{cases}$

5. $\begin{cases} \dfrac{x+a-b}{y-a-b} = \dfrac{x-b}{y-a}, \\ \dfrac{b}{x-a} = \dfrac{a}{y+b}. \end{cases}$

6. $\begin{cases} \dfrac{2n}{x+ny} - \dfrac{1}{n-ny} = 1, \\ \dfrac{10n}{x+ny} + \dfrac{3}{n-ny} = 1. \end{cases}$

7. $\begin{cases} \dfrac{ax+by}{2} + x = \dfrac{a+1}{a}, \\ \dfrac{ax+by}{2} + y = \dfrac{b+1}{b}. \end{cases}$

8. $\begin{cases} \dfrac{1}{2}(x+y) = 1 + \dfrac{x-y}{2a}, \\ \dfrac{a}{2}(x-y) = 1 + \dfrac{x-y}{2a}. \end{cases}$

9. $\begin{cases} a^2x - b^2y = 0, \\ (a^2+b^2)x + (a^2-b^2)y = a^4 + b^4. \end{cases}$

10. $\begin{cases} (a+b)x + (a-b)y = a^2 + b^2, \\ (a-b)x + (a+b)y = a^2 - b^2. \end{cases}$

11. $\begin{cases} \dfrac{xy}{x+y} = a, \\ \dfrac{xz}{x+z} = b, \\ \dfrac{yz}{y+z} = c. \end{cases}$

12. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} - \dfrac{1}{z} = a, \\ \dfrac{1}{x} - \dfrac{1}{y} + \dfrac{1}{z} = b, \\ \dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = -c. \end{cases}$

13. $\begin{cases} x + y + z = 6, \\ x + y + u = 7, \\ x + z + u = 8, \\ y + z + u = 9. \end{cases}$

14. $\begin{cases} x + y + z - u = 11, \\ x + y - z + u = 17, \\ x - y + z + u = 9, \\ -x + y + z + u = 12. \end{cases}$

15. $\begin{cases} \dfrac{2(y+5)}{5x} - \dfrac{7x}{4y+1} = \dfrac{1}{3}, \\ \dfrac{3(y+5)}{7x} - \dfrac{3x}{4y+1} = 1. \end{cases}$

16. $\begin{cases} \dfrac{y+1}{2x} + \dfrac{5x}{y} = 3\frac{1}{2}, \\ \dfrac{y+1}{3x} + \dfrac{7x}{y} = 3\frac{4}{5}. \end{cases}$

17. $\begin{cases} \dfrac{ax}{b} + \dfrac{by}{c} + \dfrac{cz}{a} = a + b + c, \\ \dfrac{cx}{b} + \dfrac{ay}{c} = a + c, \\ cy + az = a^2 + c^2. \end{cases}$

18. $\begin{cases} \dfrac{x+y}{a+b} = \dfrac{y+z}{a}, \\ \dfrac{y-x}{y+x} = \dfrac{a-b}{a+b}, \\ x + y + z = a + b. \end{cases}$

19. $$\begin{cases} \dfrac{10}{2x+3y-29}+\dfrac{9}{7x-8y+24}=\dfrac{17}{48}, \\ \dfrac{2x+3y-29}{8}=\dfrac{7x-8y}{3}+8. \end{cases}$$

20. $$\begin{cases} \frac{1}{2}(a+b-c)x+\frac{1}{2}(a-b+c)y=a^2+(b-c)^2, \\ \frac{1}{2}(a-b+c)x+\frac{1}{2}(a+b-c)y=a^2-(b-c)^2. \end{cases}$$

21. $$\begin{cases} \dfrac{x}{n^2-1}-\dfrac{y}{a^2-1}=a^2-n^2, \\ \dfrac{x}{a^2+1}+\dfrac{y}{n^2+1}=a^2+n^2-2. \end{cases}$$

22. $$\begin{cases} \dfrac{y-6}{x-4}-\dfrac{10}{16-x^2}=\dfrac{y+6}{x+4}, \\ \dfrac{5}{x^2-3x}+\dfrac{3}{3y-xy}=-\dfrac{10}{xy}. \end{cases}$$

Problems.

18. Pr. **1.** The sum of the two digits of a number is 12. If the digits are interchanged, the resulting number will exceed the original one by three-fourths of the original number. What is the number?

Let x stand for the units' digit, and y for the tens' digit.

Then the original number is $10y+x$.

When the digits are interchanged, the resulting number is $10x+y$.

The first condition of the problem states,

in *verbal* language: *the sum of the digits is* 12;

in *algebraic* language: $\quad x+y=12. \qquad (1)$

The second condition states,

in *verbal* language: *the resulting number minus the original number is equal to* $\frac{3}{4}$ *of the original number;*

in *algebraic* language: $10x+y-(10y+x)=\frac{3}{4}(10y+x). \qquad (2)$

Solving (1) and (2), $\quad x=8,\ y=4.$

Therefore the required number is 48.

Pr. **2.** A tank can be filled by two pipes. If the first is left open 6 minutes, and the second 7 minutes, the tank will be filled; or if the first is left open 3 minutes, and the second 12 minutes, the tank will be filled. In what time can each *pipe fill the* tank?

Let x stand for the number of minutes it takes the first pipe to fill the tank, and y for the number of minutes it takes the second pipe. Let the capacity of the tank be represented by 1.

Then in 1 minute the first pipe fills $\frac{1}{x}$ of the tank, and in 6 minutes $\frac{6}{x}$ of the tank; the second pipe fills $\frac{7}{y}$ of the tank in 7 minutes. Therefore, by the conditions of the problem,

$$\frac{6}{x}+\frac{7}{y}=1; \quad \frac{3}{x}+\frac{12}{y}=1.$$

Whence $$x=10\tfrac{1}{5},\ y=17.$$

Pr. 3. The sum of the three digits of a number is 9. The digit in the hundreds' place is equal to one-eighth of the number composed of the two other digits, and the digit in the units' place is equal to one-eighth of the number composed of the two other digits. What is the number?

Let x stand for the units' digit,

y for the tens' digit,

and z for the hundreds' digit.

Then, by the first condition,

$$x+y+z=9. \tag{1}$$

The number composed of the tens' and units' digits is $10y+x$. Therefore, by the second condition,

$$z=\tfrac{1}{8}(10y+x). \tag{2}$$

The number composed of the hundreds' and tens' digits is $10z+y$.

Therefore, $$x=\tfrac{1}{8}(10z+y). \tag{3}$$

Solving equations (1)–(3), we obtain,

$$x=4,\ y=2,\ z=3.$$

Therefore, the required number is 324.

Pr. 4. The report of a cannon travels 172.21 yards with the wind toward A in the same time that it travels 167.97 yards against the wind toward B. Three seconds after it is fired it is heard at A and B, which are 2041.08 yards apart. What is

the velocity of the report in still air, and what is the velocity of the wind?

Let x stand for the number of yards the report travels a second in still air,

and y for the number of yards the wind travels a second.

Then, in 1 second the report travels $x+y$ yards with the wind toward A, and $x-y$ yards against the wind toward B.

Therefore, it takes $\frac{172.21}{x+y}$ seconds to travel 172.21 yards toward A, and $\frac{167.97}{x-y}$ seconds to travel 167.97 yards toward B.

Consequently, by the first condition,

$$\frac{172.21}{x+y}=\frac{167.97}{x-y}. \tag{1}$$

In 3 seconds the report travels $3(x+y)$ yards *to* A, and $3(x-y)$ yards *to* B.

Therefore, by the second condition,

$$3(x+y)+3(x-y)=2041.08. \tag{2}$$

Solving equations (1) and (2), we obtain

$$x=340.18, \quad y=4.24.$$

Therefore, the velocity of the report in still air is 340.18 yards a second, and the velocity of the wind is 4.24 yards a second.

Pr. **5**. Two boys, A and B, run a race from P to Q and return. A, the faster runner, on his return meets B 90 feet from Q, and reaches P 3 minutes ahead of B. If he had run again to Q, he would have met B at a distance from P equal to one-sixth of the distance from P to Q. How far is Q from P, and at what rates do A and B run?

Let x stand for the number of feet from P to Q,

y for the number of feet A runs in 1 minute,

z for the number of feet B runs in 1 minute.

When they first meet 90 feet from Q, A has evidently run $x+90$ feet in $\frac{x+90}{y}$ minutes, and B has run $x-90$ feet in $\frac{x-90}{z}$ minutes.

Therefore,
$$\frac{x+90}{y}=\frac{x-90}{z}. \tag{1}$$

A runs $2x$ feet, from P to Q and return, in $\frac{2x}{y}$ minutes, and B the same distance in $\frac{2x}{z}$ minutes.

Therefore, by the second condition,
$$\frac{2x}{y}=\frac{2x}{z}-3. \tag{2}$$

If A had again met B, he would have run $2x+\frac{1}{6}x$, $=\frac{13x}{6}$, feet in $\frac{13x}{6y}$ minutes, and B would have run $2x-\frac{1}{6}x$, $=\frac{11x}{6}$, feet in $\frac{11x}{6z}$ minutes.

Therefore, by the last condition,
$$\frac{13x}{6y}=\frac{11x}{6z}, \text{ or } \frac{13}{y}=\frac{11}{z}. \tag{3}$$

Solving equations (1)–(3), we obtain
$$x=1080,\ y=130\tfrac{10}{11},\ z=110\tfrac{10}{13}.$$

Therefore the distance from P to Q is 1080 feet; A runs $130\frac{10}{11}$ feet a minute, and B runs $110\frac{10}{13}$ feet a minute.

EXERCISES VII.

1. Find two numbers whose sum is 19 and whose difference is 7.

2. If one number be multiplied by 3 and another by 7, the sum of the products will be 58; if the first be multiplied by 7 and the second by 3, the sum will be 42. What are the numbers?

3. In a meeting of 48 persons, a motion was carried by a majority of 18. How many persons voted for the motion and how many against it?

4. If one of two numbers be divided by 6 and the other by 5, the sum of the quotients will be 52; if the first be divided by 8 and the second by 12, the sum of the quotients will be 31. What are the numbers?

5. Find two numbers, such that if 1 be subtracted from the first and added to the second, the results will be equal; while if 5 be subtracted from the first and the second be subtracted from 5, these results will also be equal.

6. If 45 be subtracted from a number, the remainder will be a certain multiple of 5; but if the number be subtracted from 135, the remainder will be the same multiple of 10. What is the number, and what multiple of 5 is the first remainder?

7. If 1 be added to the numerator of a fraction, the resulting fraction will be equal to $\frac{1}{4}$; but if 1 be added to the denominator, the resulting fraction will be equal to $\frac{1}{5}$. What is the fraction?

8. A said to B: "Give me three-fourths of your marbles and I shall have 100 marbles." B said to A: "Give me one-half of your marbles and I shall have 100 marbles." How many marbles had A and B?

9. A bag contains white and black balls. One-half of the number of white balls is equal to one-third of the number of black balls, and twice the number of white balls is 6 less than the total number of balls. How many balls of each color are there?

10. The sum of two numbers is 47. If the greater be divided by the less, the quotient and the remainder will each be 5. What are the numbers?

11. A father said to his son: "After 3 years I shall be three times as old as you will be, and 7 years ago I was seven times as old as you then were." What were the ages of father and son?

12. A merchant received from one customer \$26 for 10 yards of silk and 4 yards of cloth; and from another customer \$23 for 7 yards of silk and 6 yards of cloth at the same prices. What was the price of the silk and of the cloth?

13. A merchant has two kinds of wine. If he mix 9 gallons of the poorer with 7 gallons of the better, the mixture will be worth \$1.37½ a gallon; but if he mix 3 gallons of the poorer with 5 gallons of the better, the mixture will be worth \$1.45 a gallon. What is the price of each kind of wine?

14. A man has a gold watch, a silver watch, and a chain. The gold watch and the chain cost seven times as much as the silver watch; the cost of the chain and half the cost of the silver watch is equal to three-tenths of the cost of the gold watch. If the chain cost \$40, what was the cost of each watch?

15. A and B make a purchase for \$48. A gives all of his money, and B three-fourths of his. If A had given three-fourths of his money and B all of his, they would have paid \$1.50 less. How much money had A and B?

16. A mechanic and an apprentice together receive \$40. The mechanic works 7 days and the apprentice 12 days; and the mechanic earns in 3 days \$7 more than the apprentice earns in 5 days. What wages does each receive?

17. I have 7 silver balls equal in weight and 12 gold balls equal in weight. If I place 3 silver balls in one pan of a balance and 5 gold balls in the other, I must add to the gold balls 7 ounces to maintain equilibrium. If I place in one pan 4 silver balls and in the other 7 gold balls, the balance is in equilibrium. What is the weight of each gold and of each silver ball?

18. A tank has two pumps. If the first be worked 2 hours and the second 3 hours, 1020 cubic feet of water will be discharged. But if the first be worked 1 hour and the second 2½ hours, 690 cubic feet of water will be discharged. How many cubic feet of water can each pump discharge in one hour?

19. It was intended to distribute \$25 among a certain number of the poor, each adult to receive \$2.50 and each child 75 cents. But it was found that there were 3 more adults and 5 more children than was at first supposed. Each adult was therefore given \$1.75 and each child 50 cents. How many adults and how many children were there?

20. A man ordered a wine merchant to fill two casks of different sizes with wine, one at \$1.20 and the other at \$1.50 a quart, paying \$88.50 for both casks of wine. By mistake the casks were interchanged, so that the purchaser received more of the cheaper wine and less of the dearer. The merchant therefore returned to him \$1.50. How many quarts did each cask hold?

21. A and B jointly contribute \$10,000 to a business. A leaves his money in the business 1 year and 3 months, and B his money 2 years and 11 months. If their profits are equal, how much does each contribute?

22. One boy said to another: "Give me 5 of your nuts, and I shall have three times as many as you will have left." "No," said the other, "give me 2 of your nuts, and I shall have five times as many as you will have left." How many nuts had each boy?

23. A father has two sons, one 4 years older than the other. After 2 years the father's age will be twice the joint ages of his sons; and 6 years ago his age was six times the joint ages of his sons. How old is the father and each of his sons?

24. If a number of two digits be divided by the sum of the digits, the quotient will be 7. If the digits be interchanged, the resulting number will be less than the original number by 27. What is the number?

25. A man walks 26 miles, first at the rate of 3 miles an hour, and later at the rate of 4 miles an hour. If he had walked 4 miles an hour when he walked 3, and 3 miles an hour when he walked 4, he would have gone 4 miles farther. How far would he have gone, if he had walked 4 miles an hour the whole time?

26. Two trains leave different cities, which are 650 miles apart, and run toward each other. If they start at the same time, they will meet after 10 hours; but if the first start $4\frac{1}{3}$ hours earlier than the second, they will meet 8 hours after the second train starts. What is the speed of each train?

27. If the base of a rectangle be increased by 2 feet, and the altitude be diminished by 3 feet, the area will be diminished by 48 square feet. But if the base be increased by 3 feet, and the altitude be diminished by 2 feet, the area will be increased by 6 square feet. Find the base and the altitude of the rectangle.

28. A number of three digits is in value between 400 and 500, and the sum of its digits is 9. If the digits be reversed, the resulting number will be $\frac{36}{47}$ of the original number. What is the number?

29. The report of a cannon travels with the wind 344.42 yards a second, and against the wind 335.94 yards a second. What is the velocity of the report in still air, and what is the velocity of the wind?

30. The sum of three digits of a number is 14; the sum of the first and the third digit is equal to the second; and if the digits in the units' and in the tens' place be interchanged, the resulting number will be less than the original number by 18. What is the number?

31. The sum of the ages of A, B, and C is 69 years. Two years ago B's age was equal to one-half of the sum of the ages of A and C, and 10 years hence the sum of the ages of B and C will exceed A's age by 31 years. What are the present ages of A, B, and C?

32. The total capacity of three casks is 1440 quarts. Two of them are full and one is empty. To fill the empty cask it takes all the contents of the first and one-fifth of the contents of the second, or the contents of the second and one-third of the contents of the first. What is the capacity of each cask?

33. Three brothers wished to buy a house worth \$ 70,000, but none of them had enough money. If the oldest brother had given the second brother one-third of his money, or the youngest brother one-fourth of his money, each of the latter would then have had enough money to buy the house. But the oldest brother borrowed one-half of the money of the youngest and bought the house. How much money had each brother?

34. A father's age is twenty-one times the difference between the ages of his two sons. Six years ago his age was six times the sum of his sons' ages, and two years hence it will be twice the sum of their ages. Find the ages of the father and his two sons.

35. Find the contents of three vessels from the following data: If the first be filled with water and the second be filled from it, the first will then contain two-thirds of its original contents; if from the first, when full, the third be filled, the first will then contain five-ninths of its original contents; finally, if from the first, when full, the second and third be filled, the first will then contain 8 gallons.

36. Two messengers, A and B, travel toward each other, starting from two cities which are 805 miles distant from each other. If A starts $5\frac{3}{4}$ hours earlier than B, they will meet $6\frac{1}{8}$ hours after B starts. But if B starts $5\frac{3}{4}$ hours earlier than A, they will meet $5\frac{5}{8}$ hours after A starts. At what rates do A and B travel?

37. Each of two servants was to receive \$ 160, a dress, and a pair of shoes for one year's services. One servant left after 8 months, and received the dress and \$ 106; the other servant left after $9\frac{1}{2}$ months, and received a pair of shoes and \$ 142. What was the value of the dress, and of the pair of shoes?

38. On the eve of a battle, one army had 5 men to every 6 men in the other. The first army lost 14,000 men, and the second lost 6000 men. The first army then had 2 men to every 3 men in the other. How many men were there originally in each army?

39. If the sum of two numbers, each of three digits, be increased by 1, the result will be 1000. If the greater be placed on the left of the less, and a decimal point be placed between them, the resulting number will be six times the number obtained by placing the smaller number on the left of the greater, with a decimal point between them. What are the numbers?

40. Three cities A, B, and C, are situated at the vertices of a triangle. The distance from A to C by way of B is 82 miles, from B to A by way of C is 97 miles, and from C to B by way of A is 89 miles. How far are A, B, and C from one another?

41. A regiment of 600 soldiers is quartered in a four-story building. On the first floor are twice as many men as are on the fourth; on the second and third are as many men as are on the first and fourth; and to every 7 men on the second there are 5 on the third. How many men are quartered on each floor?

42. Four men are to do a piece of work. A and B can do the work in 10 days, A and C in 12 days, A and D in 20 days, and B, C, and D in $7\frac{1}{2}$ days. In how many days can each man do the work, and in how many days can they all together do the work?

43. The year in which printing was invented is expressed by a figure of four digits, whose sum is 14. The tens' digit is one-half of the units' digit, and the hundreds' digit is equal to the sum of the thousands' and the tens' digit. If the digits be reversed, the resulting number will be equal to the original number increased by 4905. In what year was printing invented?

44. A vessel sails 110 miles with the current and 70 miles against the current in 10 hours. On a second trip, it sails 88 miles with the current and 84 miles against the current in the same time. How many miles can the vessel sail in still water in one hour, and what is the speed of the current?

45. A and B run a race of 400 yards. In the first heat A gives B a start of 20 seconds, and wins by 50 yards. In the second heat A gives B a start of 125 yards, and wins by 5 seconds. What is the speed of each runner?

46. A and B formed a partnership. A invested $ 20,000 of his own money and $ 5000 which he borrowed; B invested $ 22,000 of his own money and $ 8000 which he borrowed at the same rate of interest as was paid by A. At the end of a year, A's share in the profits amounted to $ 1750 more than the interest on his $ 5000, and B's share to $ 2000 more than the interest on his $ 8000. What rate per cent interest did they pay, and what rate per cent did they realize on their investments?

47. Two bodies move along the circumference of a circle in the same direction from two different points, the shorter distance between which, measured along the circumference, is 160 feet. One body will overtake the other in 32 seconds, if they move in one direction; or in 40 seconds, if they move in the opposite direction. While the second goes once around the circumference, the distance passed over by the first exceeds the circumference by 45 feet. What is the circumference of the circle, and at what rates do the bodies move?

48. A number of workmen, who receive the same wages, earn together a certain sum. Had there been 7 more workmen, and had each one received 25 cents more, their joint earnings would have increased by $ 18.65. Had there been 4 fewer workmen, and had each one received 15 cents less, their joint earnings would have decreased by $ 9.20. How many workmen are there, and how much does each one receive?

49. A farmer has enough feed for his oxen to last a certain number of days. If he were to sell 75 oxen, his feed would last 20 days longer. If, however, he were to buy 100 oxen, his feed would last 15 days less. How many oxen has he, and *for how many* days has he enough feed?

50. An alloy of tin and lead, weighing 40 pounds, loses 4 pounds in weight when immersed in water. Find the amount of tin and lead in the alloy, if 10 pounds of tin lose $1\frac{3}{8}$ pounds when immersed in water, and 5 pounds of lead lose .375 of a pound.

51. Two men were to receive $ 96 for a certain piece of work, which they could do together in 30 days. After half of the work was done, one of them stopped for 8 days, and then the other stopped for 4 days. They finally completed the work in $35\frac{1}{2}$ days. How many dollars should each one receive, and in what time could each one have done the work alone?

52. It took a certain number of workmen 6 hours to carry a pile of stones from one place to another. Had there been 2 more workmen, and had each one carried 4 pounds more at each trip, it would have taken them 1 hour less to complete the work. Had there been 3 fewer workmen, and had each one carried 5 pounds less at each trip, it would have taken them 2 hours longer to complete the work. How many workmen were there, and how many pounds did each one carry at every trip?

53. Three carriages travel from A to B. The second carriage travels every 4 hours 1 mile less than the first, and is 4 hours longer in making the journey. The third carriage travels every 3 hours $1\frac{3}{4}$ miles more than the second, and is 7 hours less in making the journey. How far is B from A, and how many hours does it take each carriage to make the journey?

54. A fox pursued by a dog is 60 of her own leaps ahead of the dog. The fox makes 9 leaps while the dog makes 6, but the dog goes as far in 3 leaps as the fox goes in 7. How many leaps does each make before the dog catches the fox?

CHAPTER XI.

INEQUALITIES.

1. One number is greater than a second number when the remainder obtained by subtracting the second number from the first is *positive*.

Thus, since $6 - 4, = 2$, is positive, $6 > 4$.

One number is less than a second number when the remainder obtained by subtracting the second number from the first is *negative*.

Thus, since $-5 - 2, = -7$, is negative, $-5 < +2$.

In general,

$$a > b, \text{ when } a - b \text{ is } \textit{positive},$$

and

$$a < b, \text{ when } a - b \text{ is } \textit{negative}.$$

2. An **Inequality** is a statement that two numbers or expressions are unequal; as $a^2 + b^2 > a^2$.

The members or sides of an inequality are the numbers or expressions which are connected by one of the signs of inequality, $>$ or $<$.

3. Two inequalities are of the **Same** or **Opposite Species**, or are said to subsist in the *same* or *opposite sense*, according as they have the *same* or *opposite* sign of inequality.

E.g., $8 > 3$ and $-5 > -7$ are inequalities of the same species; $0 > -1$ and $0 < 1$ are inequalities of opposite species.

Principles of Inequalities.

4. A relation of inequality between two numbers can be stated in two ways; as $7 > 3$, or $3 < 7$.

That is, *if the members of an inequality be interchanged, the sign of inequality must be reversed.*

5. *If one number be greater than a second, and this second number be greater than a third, then the first number is greater than the third;* that is,

If $a > b$ and $b > c$, then $a > c$.

In like manner, if $a < b$ and $b < c$, then $a < c$.

E.g., $3>2$, $2>1$, and $3>1$; $-3<-2$, $-2<0$, and $-3<0$.

6. An inequality will continue to be of the same species,

(i.) *When the same number is added to, or subtracted from, each member.*

(ii.) *When each member is multiplied or divided by the same* **positive** *number.*

That is, if $$a > b,$$
then $$a + n > b + n,\quad a - n > b - n;$$
and $$an > bn,\quad a \div n > b \div n;$$
wherein n is *positive*.

E.g., $8 > 4$, and $8 + 2 > 4 + 2$, $8 - 2 > 4 - 2$;
and $$8 \times 2 > 4 \times 2,\quad 8 \div 2 > 4 \div 2.$$

7. An inequality will be reversed,

(i.) *When each member is subtracted from the same number.*

(ii.) *When each member is multiplied or divided by the same* **negative** *number.*

That is, if $$a > b,$$
then $$n - a < n - b,\quad a(-n) < b(-n),\quad \frac{a}{-n} < \frac{b}{-n}.$$

E.g., $8 > 4$, and $5 - 8 < 5 - 4$, or $-3 < 1$;
$8(-2) < 4(-2)$, or $-16 < -8$; and $\frac{8}{-2} < \frac{4}{-2}$, or $-4 < -2$.

8. There is often an advantage in using the same letter with some distinguishing marks to represent different numbers in the same discussion.

Thus, with *subscripts:* a_1, a_2, a_3, etc., read *a sub-one, a sub-two, a sub-three,* etc., or simply *a one, a two, a three,* etc.

A subscript must not be confused with an exponent. Thus, a^3 stands for the product aaa; while a_3 is a notation for a single number.

Two or More Inequalities.

9. *If the corresponding members of two or more inequalities of the same species be added, the resulting inequality will be of the same species.*

That is, if $a_1 > b_1$, $a_2 > b_2$, $\cdots$, then $a_1 + a_2 \cdots > b_1 + b_2 \cdots$.

E.g., $-5 > -7$, $3 > 2$, and $-5+3 > -7+2$; or, $-2 > -5$.

10. *If all the members of two or more inequalities of the same species be positive, and if the corresponding members be multiplied together, the resulting inequality will be of the same species.*

That is, if $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, then $a_1a_2a_3 > b_1b_2b_3$, wherein a_1, b_1, a_2, b_2, a_3, b_3 are all positive.

E.g., $12 > 4$, $3 > 2$, and $12 \times 3 > 4 \times 2$, or $36 > 8$.

11. *If the members of one inequality be subtracted from, or divided by, the corresponding members of another inequality of the same species, the resulting inequality will not necessarily be of the same species.*

That is, if $$a_1 > b_1 \text{ and } a_2 > b_2,$$
then $$a_1 - a_2 \text{ may or may not } > b_1 - b_2,$$
and $$\frac{a_1}{a_2} \text{ may or may not } > \frac{b_1}{b_2}.$$

E.g., $11 > 6$, $4 > 3$, and $11 - 4 > 6 - 3$, $\frac{11}{4} > \frac{6}{3}$;

$$5 > 4,\ 3 > 1, \text{ but } 5 - 3 < 4 - 1,\ \tfrac{5}{3} < \tfrac{4}{1};$$
$$8 > 6,\ 4 > 2, \text{ while } 8 - 4 = 6 - 2;$$
$$8 > 6,\ 4 > 3, \text{ while } \tfrac{8}{4} = \tfrac{6}{3}.$$

These examples show the truth of the principle enunciated.

12. Transformation of Inequalities. — The preceding principles enable us to make the following transformations of in*equalities:*

(i.) *Any term may be transferred from one member of an inequality to the other, if its sign be reversed.*

E.g., if $a - b > c$, then $a > b + c$.

(ii.) *If the signs of both members of an inequality be reversed from + to −, or from − to +, the sign of inequality must be reversed.*

E.g., $-3 < 5$, and $3 > -5$.

13. Ex. **1.** Find one limit of the values of x, if

$$x > 5x - 10.$$

Transferring $5x$, $\quad -4x > -10.$

Dividing by -4, $\quad x < 2\frac{1}{2}.$

That is, the inequality is satisfied by all values of x less than $2\frac{1}{2}$.

Ex. **2.** Find the limits of the values of x, if

$$x - 5 < 4 - 2x, \tag{1}$$

and

$$5 - 2x > 7 - 4x. \tag{2}$$

Transferring in (1), $3x < 9$, whence $x < 3$;

Transferring in (2), $2x > 2$, whence $x > 1$.

Therefore the values of x lie between 3 and 1.

Ex. **3.** What values of x and y satisfy the inequality

$$5x + 3y > 11, \tag{1}$$

and the equality

$$3x + 5y = 13? \tag{2}$$

Multiplying (1) by 3, $\quad 15x + 9y > 33. \qquad (3)$

Multiplying (2) by 5, $\quad 15x + 25y = 65. \qquad (4)$

Subtracting (4) from (3), $\quad -16y > -32$, or $y < 2$.

Multiplying (1) by 5, $\quad 25x + 15y > 55. \qquad (5)$

Multiplying (2) by 3, $\quad 9x + 15y = 39. \qquad (6)$

Subtracting (6) from (5), $\quad 16x > 16$, or $x > 1$.

Pr. 1. A man receives from an investment an integral number of dollars a day. He calculates that if he were to receive \$6 more a day his investment would yield over \$270 a week; but that, if he were to receive \$14 less a day, his investment would not yield as much as \$270 in two weeks. How much does he receive a day from his investment?

Let x stand for the number of dollars which he receives a day.

Then, by the first condition,

$$7(x+6) > 270; \text{ whence } x > 32\tfrac{4}{7}.$$

And, by the second condition,

$$14(x-14) < 270; \text{ whence } x < 33\tfrac{2}{7}.$$

Therefore he receives \$33 a day from his investment.

EXERCISES I.

Determine one limit of the value of x in each of the following inequalities:

1. $x-8>4.$

2. $-3(x+10)>-20$

3. $\dfrac{3x-8}{4}-x<\dfrac{37-2x}{3}+9.$

4. $\dfrac{11a-x}{4a+b}>\dfrac{a-x}{b-a}.$

5. $x-\dfrac{a}{1-a}<1-\dfrac{x-1}{a-1}.$

6. $\dfrac{x}{a+b}+\dfrac{x}{a-b}<2a.$

Determine the limits of the values of x in each of the following systems of inequalities:

7. $\begin{cases} 6x+1>0, \\ 25-4x>0. \end{cases}$

8. $\begin{cases} \frac{1}{8}x-\frac{1}{4}x+\frac{1}{2}x>x-5, \\ \frac{1}{8}(x+2)>-\frac{1}{7}(x-2). \end{cases}$

Determine the limits of the values of x and y in each of the following systems:

9. $\begin{cases} 2x+3y=-4, \\ x-y>2. \end{cases}$

10. $\begin{cases} 7x+y=15, \\ 3x-2y>14. \end{cases}$

11. What integers have each the property that one-half of the integer, increased by 5, is greater than four-thirds of it, *diminished* by 3?

12. What integers have each the property that, if 9 be subtracted from three times the integer, the remainder will be less than twice the integer, increased by 12?

13. A has three times as much money as B. If B gives A \$10, then A will have more than seven times as much as B will have left. What are the possible amounts of money which A and B have?

Identical Inequalities.

14. Many inequalities hold for all values of the literal numbers involved; as $a^2 + b^2 > a^2$.

Such inequalities are analogous to identical equations.

15. Prove that if a is not equal to b, then $a^2 + b^2 > 2ab$.

We have $$(a - b)^2 > 0, \tag{1}$$

since the square of any positive or negative number is positive, and therefore greater than 0.

From (1), $$a^2 - 2ab + b^2 > 0;$$

whence $$a^2 + b^2 > 2ab, \text{ by Art. } \mathbf{12} \text{ (i.).}$$

EXERCISES II.

Prove the following inequalities, in which the literal numbers are all positive and unequal:

1. $a^2 + b^2 + c^2 > ab + ac + bc$.
2. $a^2b^2 + b^2c^2 + a^2c^2 > abc(a + b + c)$.
3. $ab(a + b) + bc(b + c) + ac(a + c) > 6abc$.
4. If $l^2 + m^2 + n^2 = 1$, and $l_1^2 + m_1^2 + n_1^2 = 1$, then $$ll_1 + mm_1 + nn_1 < 1.$$
5. $a^3 + b^3 > a^2b + ab^2$.
6. $a^4 + b^4 > a^3b + ab^3$.
7. $(a + b)(b + c)(c + a) > 8abc$.
8. $3(a^2 + b^2 + c^2) > (a + b + c)^2$.

CHAPTER XII.

INDETERMINATE LINEAR EQUATIONS.

1. It was shown in Ch. X., Art. 1, that the linear equation in two unknown numbers

$$x + y = 5$$

is satisfied by an *indefinite* number of sets of values of x and y.

An **Indeterminate Equation** is an equation which, like the above, has an indefinite number of solutions.

Evidently the number of solutions will be more limited if only *positive integral* values of the unknown numbers are admitted.

In this chapter we shall consider a simple method of solving in *positive integers* linear indeterminate equations.

2. Ex. 1. Solve $4x + 7y = 94$, in positive integers.

Solving for x, which has the smaller coefficient, we obtain

$$x = \frac{94 - 7y}{4} = 23 - y + \frac{2 - 3y}{4}, \qquad (1)$$

or
$$x - 23 + y = \frac{2 - 3y}{4}.$$

Since x and y are to be integers, $\frac{2 - 3y}{4}$ must be an integer. That is, y must have such a value that $2 - 3y$ shall be divisible by 4.

Let
$$\frac{2 - 3y}{4} = m, \text{ an integer.}$$

Then $y = \frac{2 - 4m}{3}$, an inconvenient form from which to determine integral values of y. But since the expression $\frac{2 - 3y}{4}$ is to be an integer, any multiple of it will be an integer. We *therefore* multiply its numerator by the least number which

will make the coefficient of y one more than a multiple of the denominator, *i.e.*, by 3.

We then have

$$\frac{6-9y}{4} = 1 - 2y + \frac{2-y}{4}, \text{ an integer.}$$

Therefore, $$\frac{2-y}{4} = m, \text{ an integer.}$$

Whence $$y = 2 - 4m. \qquad (2)$$

Then, from (1) and (2), $x = 20 + 7m.$ (3)

Any integral value of m will give to x and y integral values.

But since y is to be *positive*, $m < 1$;

and, since x is to be *positive*, $m > -3$.

Therefore the only admissible values of m are 0, -1, -2.

When $m = 0$, $x = 20$, $y = 2$;

$m = -1$, $x = 13$, $y = 6$;

$m = -2$, $x = 6$, $y = 10$.

3. An **Indeterminate System** is a system of equations which has an *indefinite* number of solutions.

Thus, if the system
$$x + y - z = 9,$$
$$2x - y + 7z = 33,$$
be solved for x and y, we obtain
$$x = 14 - 2z,\ y = 3z - 5.$$

In these values of x and y we may assign any value to z and obtain corresponding values of x and y.

4. In solving a system of *two* linear equations in *three* unknown numbers, we first eliminate one of the unknown numbers, and apply to the resulting equation the preceding method.

Pr. A party of 20 people, consisting of men, women, and children, pay a hotel bill of \$67. Each man pays \$5, each woman \$4, and each child \$1.50. How many of the company are men, how many women, and how many children?

Let x stand for the number of men, y for the number of women, z for the number of children.

Then, by the conditions of the problem,

$$x + y + z = 20, \qquad (1)$$

$$5x + 4y + \tfrac{3}{2}z = 67. \qquad (2)$$

Eliminating z, $\qquad 7x + 5y = 74.$

Solving this equation, we obtain

$$x = 2 - 5m,\ y = 12 + 7m,\ z = 6 - 2m.$$

When $m = 0, \qquad x = 2,\ y = 12,\ z = 6;$

$m = -1, \qquad x = 7,\ y = 5,\ z = 8.$

EXERCISES.

Solve in positive integers:

1. $5x+8y=29$. **2.** $3x+5y=10$. **3.** $12x+13y=175$.

4. $25x+15y=215$. **5.** $5x+13y=229$. **6.** $34x+89y=407$.

7. $\begin{cases} x + 3y + 5z = 44, \\ 3x + 5y + 7z = 68. \end{cases}$ **8.** $\begin{cases} 8x + 3y - 2z = 8, \\ 7x - 2y - z = 8. \end{cases}$

Solve in least positive integers:

9. $89x-144y=1$. **10.** $14x-49y=133$. **11.** $67x-43y=5$.

12. Divide 1000 into two parts so that one part shall be a multiple of 13, and the other a multiple of 53.

13. What positive integers when divided by 4 give a remainder 3, and when divided by 5 give a remainder 4?

14. A farmer received \$16 for a number of turkeys and chickens. If he was paid \$2 for each turkey and \$.75 for each chicken, how many of each did he sell?

15. A gardener has fewer than 1000 trees. If he plants them in rows of 37 each, he will have 8 left; but if he plants them in a different number of rows of 43 each, he will have 11 left. How many trees has he?

16. A said to B: "If I had eight times as much money as I now have, and you had seven times as much money as you now have, and I were to give you \$1, we should have equal amounts." How many dollars had each?

CHAPTER XIII.

INVOLUTION

1. Involution is the process of raising a number to any required power.

Powers of Powers.

2. Ex. 1. $(a^4)^5 = a^4a^4a^4a^4a^4 = a^{4+4+4+4+4} = a^{4\times 5} = a^{20}.$

Ex. 2.
$$\begin{aligned}(x^9)^{10} &= x^9x^9x^9 \cdots \text{to 10 factors}\\ &= x^{9+9+9+\cdots\text{ to 10 summands}} = x^{9\times 10} = x^{90}.\end{aligned}$$

These examples illustrate the following method of finding any required power of a given power:

Multiply the exponent of the given power by the exponent of the required power; or, stated symbolically,

$$(a^m)^n = a^{mn}.$$

For,
$$\begin{aligned}(a^m)^n &= a^ma^ma^m \cdots \text{to } n \text{ factors}\\ &= a^{m+m+m+\cdots\text{ to } n \text{ summands}} = a^{mn}.\end{aligned}$$

Powers of Products.

3. Ex. 1.
$$\begin{aligned}(ab)^4 &= (ab)(ab)(ab)(ab)\\ &= (aaaa)(bbbb) = a^4b^4.\end{aligned}$$

Ex. 2
$$\begin{aligned}(xy)^{10} &= (xy)(xy)(xy) \cdots \text{to 10 factors}\\ &= (xxx \cdots \text{to 10 factors})(yyy \cdots \text{to 10 factors})\\ &= x^{10}y^{10}.\end{aligned}$$

These examples illustrate the following method of finding any required power of a product:

Take the product of the factors, each raised to the required power; or, stated symbolically,

$$(ab)^n = a^nb^n; \quad (abc)^n = a^nb^nc^n; \text{ etc.}$$

For, $$\begin{aligned}(ab)^n &= (ab)(ab)(ab)\cdots \text{to } n \text{ factors}\\ &= (aaa\cdots \text{to } n \text{ factors})(bbb\cdots \text{to } n \text{ factors})\\ &= a^nb^n.\end{aligned}$$

In like manner, $(abc)^n = a^nb^nc^n$; and so on.

4. The converse of the principle of Art. 3 is evidently true. That is,

$$a^mb^m = (ab)^m;\ a^mb^mc^m = (abc)^m;\ \text{etc.}$$

5. The principles of Arts. 2–3 prove the method, already given in Ch. V., Art. 5, of raising a monomial to any required power.

Raise the numerical coefficient to the required power, and multiply the exponent of each literal factor by the exponent of the required power.

Ex. 1. $$(4\,a^3b)^2 = 4^2\,a^{3\times 2}b^2 = 16\,a^6b^2.$$

Ex. 2. $$(-3\,a^4x^2)^3 = (-3)^3\,a^{4\times 3}x^{2\times 3} = -27\,a^{12}x^6.$$

Powers of Fractions.

6. Ex. 1. $$\left(\frac{2\,x^2}{y^3}\right)^2 = \frac{2\,x^2}{y^3}\times\frac{2\,x^2}{y^3} = \frac{(2\,x^2)^2}{(y^3)^2} = \frac{4\,x^4}{y^6}.$$

Ex. 2. $$\begin{aligned}\left(\frac{a}{b}\right)^9 &= \frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\cdots \text{to 9 factors}\\ &= \frac{aaa\cdots \text{to 9 factors}}{bbb\cdots \text{to 9 factors}} = \frac{a^9}{b^9}.\end{aligned}$$

These examples illustrate the following method of raising any fraction to a required power:

Raise each term of the fraction to the required power; or, stated symbolically,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

For, $$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\cdots \text{to } n \text{ factors}\\ &= \frac{aaa\cdots \text{to } n \text{ factors}}{bbb\cdots \text{to } n \text{ factors}} = \frac{a^n}{b^n}.\end{aligned}$$

EXERCISES I.

Write the cubes and the fourth powers of:

1. x^2. **2.** $-x^4$. **3.** $2x^7$. **4.** $-3ab$.

5. $5ab^2$. **6.** $4x^2y^3$. **7.** $2m^2xy^5$. **8.** $5a^2b^5c^6$.

9. $\dfrac{a}{b}$. **10.** $\dfrac{2a}{y^2}$. **11.** $-\dfrac{3x^2}{2y^3}$. **12.** $-\dfrac{4x^2y}{3ab^3}$.

Write the squares, the cubes, and the nth powers of:

13. a^{m+1}. **14.** x^{m-2}. **15.** $2x^{m+n}y$. **16.** $-3a^{m+n-1}y^3$.

Find the values of each of the following powers:

17. $(-3x^2y^4)^3$. **18.** $(5a^5b^6c)^2$. **19.** $(-4x^4y^2z^5)^3$.

20. $(2xy^2z^3)^4$. **21.** $(-a^2xy^4)^6$. **22.** $(-2m^2n^3)^5$.

23. $\left(\dfrac{3a^2b}{4c^2d^2}\right)^3$. **24.** $\left(-\dfrac{3a^2b^5}{4m^2n^3}\right)^3$. **25.** $\left(-\dfrac{a^2bc^3}{2xy^3z}\right)^4$.

Powers of Binomials.

7. By actual multiplication, we obtain

$$(a+b)^3 = (a^2+2ab+b^2)(a+b) = a^3+3a^2b+3ab^2+b^3,$$

$$(a-b)^3 = (a^2-2ab+b^2)(a-b) = a^3-3a^2b+3ab^2-b^3,$$

$$\begin{aligned}(a+b)^4 &= (a^2+2ab+b^2)(a^2+2ab+b^2)\\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4,\end{aligned}$$

$$(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4.$$

The result of performing the indicated operation in a power of a binomial is called the **Expansion** of that power of the binomial.

In the preceding expansions the following laws are evident:

(i.) *The number of terms exceeds the binomial exponent by* 1.

(ii.) *The exponent of* $\boldsymbol{a}$ *in the first term is equal to the binomial exponent, and decreases by* 1 *from term to term.*

(iii.) *The exponent of* $\boldsymbol{b}$ *in the second term is* 1 *and increases by* 1 *from term to term, and in the last term is equal to the binomial exponent.*

(iv.) *The coefficient of the first term is* 1, *and that of the second term, except for sign, is equal to the binomial exponent.*

(v.) *The coefficient of any term after the second is obtained, except for sign, by multiplying the coefficient of the preceding term by the exponent of* **a** *in that term, and dividing the product by a number greater by* 1 *than the exponent of* **b** *in that term.*

E.g., the coefficient of the fourth term in the expansion of

$$(a+b)^4 \text{ is } 6 \times 2 \div 3, = 4.$$

(vi.) *The signs of the terms are all positive when the terms of the binomial are both positive; the signs of the terms alternate,* + *and* −, *when one of the terms of the binomial is negative.*

Observe, as a check:

(vii.) *The sum of the exponents of* **a** *and* **b** *in any term is equal to the binomial exponent.*

(viii.) *The coefficients of two terms equally distant from the beginning and the end of the expansion are equal.*

In a subsequent chapter the above laws will be proved to hold for any positive integral power of the binomial.

8. Ex. 1.

$$\begin{aligned}(2a-3b)^4 &= (2a)^4 - 4(2a)^3(3b) + 6(2a)^2(3b)^2 \\ &\quad - 4(2a)(3b)^3 + (3b)^4 \\ &= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4.\end{aligned}$$

Ex. 2.

$$\begin{aligned}(x+2y)^5 &= x^5 + 5x^4(2y) + 10x^3(2y)^2 \\ &\quad + 10x^2(2y)^3 + 5x(2y)^4 + (2y)^5 \\ &= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5.\end{aligned}$$

EXERCISES II.

Raise each of the following expressions to the required power:

1. $(x+1)^3$.	**2.** $(a-3)^3$.	**3.** $(2x+3)^3$.
4. $(5-2y)^3$.	**5.** $(2ab+3)^3$.	**6.** $(5x-6y)^3$.
7. $(x^2-8)^3$.	**8.** $(5x^2-3y)^3$.	**9.** $(6x^2-5y^2)^3$.
10. $(x-1)^4$.	**11.** $(2x+3)^4$.	**12.** $(3x-2y)^4$.
13. $(a+b)^5$.	**14.** $(2m-3n)^5$.	**15.** $(x-y)^6$.

Powers of Multinomials.

9. We have

$$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2.$$

Therefore $\quad \boldsymbol{(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc.}$

In like manner,

$$\boldsymbol{(a+b-c)^2 = a^2+b^2+c^2+2ab-2ac-2bc.}$$
$$\boldsymbol{(a-b-c)^2 = a^2+b^2+c^2-2ab-2ac+2bc.}$$

By repeated application of this principle we can obtain the square of a multinomial of any number of terms. We have

$$(a+b+c+d)^2 = [(a+b+c)^2 + d]^2$$
$$= a^2+b^2+c^2+2ab+2ac+2bc+2(a+b+c)d+d^2$$
$$= a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

That is, *the square of a multinomial is equal to the sum of the squares of the terms, plus the algebraic sum of twice the product of each term by each term which follows it.*

Ex. 1. $(3x+5y-7z)^2 = (3x)^2+(5y)^2+(-7z)^2+2(3x)(5y)$
$$+2(3x)(-7z)+2(5y)(-7z)$$
$$=9x^2+25y^2+49z^2+30xy-42xz-70yz.$$

EXERCISES III.

Raise each of the following expressions to the required power:

1. $(a+b+1)^2$. **2.** $(x-y-1)^2$.

3. $(2a+3b+1)^2$. **4.** $(3a-4b+5c)^2$.

5. $(a^2+a+1)^2$. **6.** $(x^2-x+1)^2$.

7. $(x^2+xy+y^2)^2$. **8.** $(a^2-3ab+b^2)^2$.

9. $(a+b+c)^3$. **10.** $(a-b-c)^3$.

11. $(a^2-a+1)^3$. **12.** $(2a-b+5)^3$.

13. $(a+b+c+d)^2$. **14.** $(a-b-c+d)^2$.

15. $(a^3-a^2+a-1)^2$. **16.** $(x^3+2x^2-3x+4)^2$.

CHAPTER XIV.

EVOLUTION.

1. A **Root** of a number is one of the equal factors of the number.

E.g., 2 is a root of 4, of 8, of 16, etc.

2. A **Second,** or **Square Root** of a number is one of *two* equal factors of the number.

E.g., since $5 \times 5 = 25$ and $(-5)(-5) = 25$, therefore $+5$ and -5 are square roots of 25.

A **Third,** or **Cube Root** of a number is one of *three* equal factors of the number.

E.g., since $3 \times 3 \times 3 = 27$, therefore 3 is a cube root of 27; since $(-3)(-3)(-3) = -27$, therefore -3 is a cube root of -27.

In general, the *qth root* of a number is one of q equal factors of the number.

E.g., a qth root of x^q is x.

3. The **Radical Sign,** $\sqrt{\ }$, is used to denote a root, and is placed before the number whose root is to be found.

The **Radicand** is the number whose root is required.

The **Index** of a root is the number which indicates what root is to be found, and is written over the radical sign. The index 2 is usually omitted.

E.g., $\sqrt[2]{9}$, or $\sqrt{9}$, denotes a second, or square root of 9; the radicand is 9, and the index is 2.

4. A vinculum is often used in connection with the radical sign to indicate what part of an expression following the sign *is affected by it.*

E.g., $\sqrt{}9 + 16$ means the sum of $\sqrt{9}$ and 16, while $\sqrt{9+16}$ means a square root of the sum $9 + 16$. Likewise $\sqrt[3]{}a^3 \times b^6$ means the product of $\sqrt[3]{a^3}$ and b^6, while $\sqrt[3]{a^3 \times b^6}$ means a cube root of a^3b^6.

Parentheses may be used instead of the vinculum in connection with the radical sign; as $\sqrt{}(9 + 16)$ for $\sqrt{9 + 16}$.

5. It follows from the definition of a root that the square of a square root of a number is the number, the cube of a cube root of a number is the number, and so on.

E.g., $(\sqrt{4})^2 = 4$; $(\sqrt[3]{8})^3 = 8$; etc.

In general, $(\sqrt[q]{a})^q = a.$

6. An **Even Root** is one whose *index* is *even;* as $\sqrt{a^2}$, $\sqrt[4]{a^4}$, $\sqrt[2q]{a^{2q}}$.

An **Odd Root** is one whose *index* is *odd;* as $\sqrt[3]{8}$, $\sqrt[5]{8^{10}}$, $\sqrt[2q+1]{a^{2q+1}}$.

7. In this chapter we shall consider only roots of powers whose exponents are multiples of the indices of the required roots; as $\sqrt{16}$, $=\sqrt{4^2}$, $\sqrt[3]{a^3}$, $\sqrt[q]{a^{kq}}$.

Number of Roots.

8. Since $(\pm 4)^2 = 16$, therefore $\sqrt{16} = \pm 4$;

since $(\pm a)^4 = a^4$, therefore $\sqrt[4]{a^4} = \pm a$.

These examples illustrate the principle:

A positive number has at least two even roots, equal and opposite; i.e., *one positive and one negative.*

9. Since $(-3)^3 = -27$, therefore $\sqrt[3]{-27} = -3$;

since $2^5 = 32$, therefore $\sqrt[5]{32} = 2$.

These examples illustrate the principle:

A positive or a negative number has at least one odd root of the same sign as the number itself.

10. *Since* $(+4)^2 = +16$ *and* $(-4)^2 = +16$, *there is no number, with which we are as yet familiar*, whose square is -16.

Consequently $\sqrt{-16}$ cannot be expressed as a positive or as a negative number; that is, in terms of the numbers as yet used in this book.

Such roots are called **Imaginary Numbers**, and will be considered in Ch. XVI.

Evolution.

11. Evolution is the process of finding a root of a given number.

12. *In the following articles the radicands are limited to positive values, and the roots to positive roots.*

13. (i.) Since $(a^2)^3 = a^6$, therefore $\sqrt[3]{a^6} = a^2 = a^{\frac{6}{3}}$.

This example illustrates the principle:

The root of a power is obtained by dividing the exponent of the power by the index of the root.

E.g., $\sqrt[4]{a^4} = a$; $\sqrt[5]{a^{15}} = a^{\frac{15}{5}} = a^3$.

In general, $\sqrt[q]{a^{nq}} = a^{\frac{nq}{q}} = a^n$.

For, since $(a^n)^q = a^{nq}$, therefore $\sqrt[q]{a^{nq}} = a^n = a^{\frac{nq}{q}}$.

(ii.) Since $(ab)^2 = a^2b^2$, therefore $\sqrt{(a^2b^2)} = ab = \sqrt{a^2} \times \sqrt{b^2}$.

This example illustrates the principle:

The root of a product of two or more factors is equal to the product of the like roots of the factors, and conversely.

E.g., $\sqrt{(16 \times 25)} = \sqrt{16} \times \sqrt{25} = 4 \times 5 = 20$;

$\sqrt[3]{(8a^3b^6)} = \sqrt[3]{8} \times \sqrt[3]{a^3} \times \sqrt[3]{b^6} = 2 \times a \times b^2 = 2ab^2$.

In general, $\sqrt[q]{(a^qb^q)} = \sqrt[q]{a^q} \times \sqrt[q]{b^q}$.

For, since $(ab)^q = a^qb^q$, therefore $\sqrt[q]{(a^qb^q)} = ab = \sqrt[q]{a^q}\sqrt[q]{b^q}$.

(iii.) Since $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, therefore $\sqrt{\frac{a^2}{b^2}} = \frac{a}{b} = \frac{\sqrt{a^2}}{\sqrt{b^2}}$

This example illustrates the principle:

The root of a quotient of two numbers is equal to the quotient of the like roots of the numbers, and conversely.

E.g., $\sqrt{\dfrac{25}{16}}=\dfrac{\sqrt{25}}{\sqrt{16}}=\dfrac{5}{4}$; $\sqrt[3]{\dfrac{27\,a^3}{b^6}}=\dfrac{\sqrt[3]{(27\,a^3)}}{\sqrt[3]{b^6}}=\dfrac{3\,a}{b^2}$.

In general, $$\sqrt[q]{\frac{a^q}{b^q}}=\frac{\sqrt[q]{a^q}}{\sqrt[q]{b^q}}.$$

For, since $\left(\dfrac{a}{b}\right)^q=\dfrac{a^q}{b^q}$, therefore $\sqrt[q]{\dfrac{a^q}{b^q}}=\dfrac{a}{b}=\dfrac{\sqrt[q]{a^q}}{\sqrt[q]{b^q}}$.

Roots of Monomials.

14. The *positive* root of a positive number can be found by applying the principles of Art. 13.

The *negative even* root of a positive number is found by prefixing the negative sign to its positive root.

Since $\sqrt[3]{-8}=-2$, and $-\sqrt[3]{8}=-2$,

therefore $$\sqrt[3]{-8}=-\sqrt[3]{8}.$$

That is, the *negative odd* root of a negative number is found by prefixing the negative sign to the positive root of the radicand taken positively.

Ex. **1.** $$\sqrt{(16\,a^2b^4)}=\sqrt{16}\times\sqrt{a^2}\times\sqrt{b^4}=4\,a^{\frac{2}{2}}b^{\frac{4}{2}}$$
$$=4\,ab^2\text{, the positive square root.}$$

Therefore $\pm\sqrt{(16\,a^2b^4)}=\pm\,4\,ab^2$.

In the following examples we shall give only the *positive* even roots.

Ex. **2.** $$\sqrt[3]{(-27\,x^3y^6z^9)}=\sqrt[3]{-27}\times\sqrt[3]{x^3}\times\sqrt[3]{y^6}\times\sqrt[3]{z^9}$$
$$=-3\,x^{\frac{3}{3}}y^{\frac{6}{3}}z^{\frac{9}{3}}=-3\,xy^2z^3.$$

These examples illustrate the following method:

Take the required root of the numerical coefficient, and divide the exponent of each literal factor by the index of the required root.

Ex. **3.** $$\sqrt[4]{\frac{16\,a^8b^{12}}{625\,c^{16}}}=\frac{\sqrt[4]{(16\,a^8b^{12})}}{\sqrt[4]{(625\,c^{16})}}=\frac{\sqrt[4]{16}\,a^{\frac{8}{4}}b^{\frac{12}{4}}}{\sqrt[4]{625}\,c^{\frac{16}{4}}}=\frac{2\,a^2b^3}{5\,c^4}.$$

15. It is frequently of advantage to separate a number expressed in figures into its prime factors before taking the root.

Ex. **4.** $\sqrt{(15 \times 40 \times 216)} = \sqrt{(5 \cdot 3 \times 2^3 \cdot 5 \times 2^3 \cdot 3^3)}$
$$= \sqrt{(5^2 \cdot 3^4 \cdot 2^6)} = 5 \cdot 3^2 \cdot 2^3 = 360.$$

EXERCISES I.

Simplify the following expressions:

1. $\sqrt{x^{10}}$. **2.** $\sqrt[3]{-a^9}$. **3.** $\sqrt[4]{x^{12}}$. **4.** $\sqrt[5]{a^{10n}}$.

5. $\sqrt{(36\,x^2)}$. **6.** $\sqrt[3]{(27\,y^3)}$. **7.** $\sqrt[3]{(-64\,z^6)}$.

8. $\sqrt[4]{(81\,x^{12})}$. **9.** $\sqrt[5]{(32\,a^{10})}$. **10.** $\sqrt{(16\,a^2x^6)}$.

11. $\sqrt[3]{(-8\,m^6n^9)}$. **12.** $\sqrt[4]{(16\,a^8y^4)}$. **13.** $\sqrt[5]{(-243\,a^5b^{15})}$.

14. $\sqrt{(6\tfrac{1}{4}\,a^6b^{4n-2})}$. **15.** $\sqrt[4]{(5\tfrac{1}{16}\,x^{4n}y^{8n-12})}$. **16.** $\sqrt{[81\,a^4(a^2+x^2)^6]}$.

17. $\sqrt{(3\,ax^{3n} \times 27\,a^3x^{3n})}$. **18.** $\sqrt[3]{(9\,a^4y^{2n} \times 3\,a^2y^n)}$.

19. $\sqrt{\dfrac{49\,a^{10}}{b^4c^6}}$. **20.** $\sqrt[3]{-\dfrac{a^{21}x^{15}}{343}}$. **21.** $\sqrt[3]{\dfrac{27\,a^3b^6}{64\,x^9y^{12}}}$.

22. $\sqrt{\dfrac{9\,a^6b^{4m}}{c^{10}d^{2n}}}$. **23.** $\sqrt{\dfrac{625\,x^4y^{12}}{a^8b^{16}}}$. **24.** $\sqrt[3]{\dfrac{.064\,a^{12}}{b^3x^{15n}}}$.

Find the values of each of the following expressions:

25. $\sqrt{64^3}$. **26.** $\sqrt{49^5}$. **27.** $\sqrt[3]{216^2}$. **28.** $\sqrt[3]{-27^4}$.

29. $\sqrt{(40 \times 15 \times 6)}$. **30.** $\sqrt{(56 \times 40 \times 35)}$.

31. $\sqrt{1024}$. **32.** $\sqrt{2025}$. **33.** $\sqrt{12544}$.

34. $\sqrt[3]{(6 \times 20 \times 225)}$. **35.** $\sqrt[3]{(84 \times 18 \times 49)}$.

36. $\sqrt{(45\,xy \times 35\,xz \times 63\,yz)}$.

37. $\sqrt[3]{(36\,a^2bc \times 75\,ab^2c^2 \times 80\,a^3b^3)}$.

SQUARE ROOTS OF MULTINOMIALS

16. The square root of a trinomial which is the square of a binomial can be found by inspection (Ch. VI., Art. 9).

17. Since $(a + b)^2 = a^2 + 2\,ab + b^2,$

we have $\sqrt{(a^2 + 2\,ab + b^2)} = a + b.$

From this identity we infer:

(i.) *The first term of the root is the square root of the first term of the trinomial;* i.e., $a = \sqrt{a^2}$.

(ii.) *If the square of the first term of the root be subtracted from the trinomial, the remainder will be*

$$2ab + b^2, \ = (2a + b)b.$$

Twice the first term of the root, $2a$, is called the **Trial Divisor.**

(iii.) *The second term of the root is obtained by dividing the first term of the remainder by the trial divisor;* i.e., $b = \dfrac{2ab}{2a}$.

The trial divisor plus the second term of the root is called the **Complete Divisor.**

(iv.) *If the product of the complete divisor by the second term of the root be subtracted from the first remainder, the second remainder will be* 0.

The work may be arranged as follows:

$$\begin{array}{r|ll}
a^2 + 2ab + b^2 & \underline{a + b} & \\
\underline{a^2 \qquad\qquad} & 2a & \text{trial divisor} \\
2ab & 2ab \div 2a = b, & \text{second term of root} \\
 & 2a + b & \text{complete divisor} \\
\underline{2ab + b^2} & \underline{= (2a + b)b} &
\end{array}$$

18. Ex. 1. Find the square root of $4x^4 - 12x^2y + 9y^2$.

The work, arranged as above, writing only the trial and the complete divisor, is:

$$\begin{array}{r|l}
4x^4 - 12x^2y + 9y^2 & \underline{2x^2 - 3y} \\
\underline{4x^4 \qquad\qquad\qquad} & 4x^2 \\
-12x^2y \qquad & \\
\underline{-12x^2y + 9y^2} & \underline{4x^2 - 3y}
\end{array}$$

The square root of $4x^4$ is $2x^2$, the first term of the root. The trial divisor is $2(2x^2), = 4x^2$. The second term of the root is $-\dfrac{12x^2y}{4x^2}, = -3y$. The complete divisor is $4x^2 - 3y$.

Ex. 2. Find the square root of

$$4x^4 - 12x^3 + 29x^2 - 30x + 25.$$

The work follows:

$$\begin{array}{l|l}
4x^4 - 12x^3 + 29x^2 - 30x + 25 & \underline{2x^2 - 3x + 5} \\
\underline{4x^4} & 4x^2 \\
\quad -12x^3 & \\
\quad \underline{-12x^3 + 9x^2} & 4x^2 - 3x \\
\qquad\qquad 20x^2 & \\
\qquad\qquad \underline{20x^2 - 30x + 25} & \underline{4x^2 - 6x + 5}
\end{array}$$

Only the trial divisor and the complete divisor of each stage are written, the other steps being performed mentally.

The square root of $4x^4$ is $2x^2$, the first term of the root. The trial divisor is $2(2x^2), = 4x^2$. The second term of the root is $-\dfrac{12x^3}{4x^2}, = -3x$. The complete divisor is $4x^2 - 3x$, which is multiplied by the second term of the root, giving $-12x^3 + 9x^2$.

The first term of the second remainder is $20x^2$.

The third term of the root is $\dfrac{20x^2}{4x^2}, = 5$.

To form the complete divisor at this stage, we multiply the part of the root previously found, $2x^2 - 3x$, by 2, and to the product add the term just found. We thus obtain $4x^2 - 6x + 5$. This complete divisor we multiply by the last term of the root.

In the preceding examples the terms were arranged to descending powers of x. They could equally well have been arranged to ascending powers.

19. The preceding method can be extended to find square roots which are multinomials of any number of terms.

The work consists of repetitions of the following steps:

After one or more terms of the root have been found, obtain each succeeding term, by dividing the first term of the remainder at that stage by twice the first term of the root.

Find the next remainder by subtracting from the last remainder the expression $(2a+b)b$, *wherein* a *stands for the part of the root already found, and* b *for the term last found.*

EXERCISES II.

Find the square root of each of the following expressions:

1. $x^4 - 4x^3 + 8x + 4.$
2. $4m^4 - 4m^3 + 5m^2 - 2m + 1.$
3. $x^4 - 2x^3 + 3x^2 - 2x + 1.$
4. $4x^4 + 12x^3 + 5x^2 - 6x + 1.$
5. $9x^4 + 12x^3 - 26x^2 - 20x + 25.$
6. $4x^4 - 28x^3 + 51x^2 - 7x + \frac{1}{4}.$
7. $x^4y^4 - 4x^3y^3 + 6x^2y^2 - 4xy + 1.$
8. $\frac{1}{9}x^4 + \frac{4}{3}x^3y + 2x^2y^2 - 12xy^3 + 9y^4.$
9. $x^4 - 6ax^3 + 13a^2x^2 - 12a^3x + 4a^4.$
10. $4a^2 + 9b^2 + 16c^2 - 12ab + 16ac - 24bc.$
11. $49x^8 + 42x^6 - 19x^4 - 12x^2 + 4.$
12. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4.$
13. $a^4 + 4a^3 + 4a^2 + 2a + 4 + \dfrac{1}{a^2}.$
14. $9a^4 + 30a^3b + 49a^2b^2 + 40ab^3 + 16b^4.$
15. $89a^2b^2 - 70ab^3 + 16a^4 - 56a^3b + 25b^4.$
16. $4a^6 - 12a^4b - 28a^3b^3 + 9a^2b^2 + 42ab^4 + 49b^6.$
17. $\dfrac{x^4}{y^4} - \dfrac{4x^3}{y} + 4x^2y^2 + 6x - 12y^3 + 9\dfrac{y^4}{x^2}.$
18. $x^4 + \dfrac{2x^3}{a} + \dfrac{x^2}{a^2} + 2ax + 2 + \dfrac{a^2}{x^2}.$
19. $1 + 2x - x^2 + 3x^4 - 2x^5 + x^6.$
20. $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6.$
21. $1 - 4a + 64a^6 - 64a^5 - 32a^3 + 48a^4 + 12a^2.$
22. $4a^6 + 17a^2 - 22a^3 + 13a^4 - 24a - 4a^5 + 16.$
23. $9x^6 + 6x^5y + 43x^4y^2 + 2x^3y^3 + 45x^2y^4 - 28xy^5 + 4y^6.$
24. $x^4 + 4x^3 + 6x^2 + 5x + 5 + \dfrac{5}{x} + \dfrac{9}{4x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4}.$

CUBE ROOTS OF MULTINOMIALS.

20. The process of finding the cube root of a multinomial is the inverse of the process of cubing the multinomial.

Since
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$= a^3 + (3a^2 + 3ab + b^2)b,$$
we have
$$\sqrt[3]{(a^3 + 3a^2b + 3ab^2 + b^3)} = a + b.$$

From the identity (2), we infer:

(i.) *The first term of the root is the cube root of the first term of the multinomial;* i.e., $a = \sqrt[3]{a^3}$.

(ii.) *If the cube of the first term of the root be subtracted from the multinomial, the remainder will be*
$$3a^2b + 3ab^2 + b^3, = (3a^2 + 3ab + b^2)b.$$

Three times the square of the first term of the root, $3a^2$, is called the **Trial Divisor.**

(iii.) *The second term of the root is obtained by dividing the first term of the remainder by the trial divisor;* i.e., $b = \dfrac{3a^2b}{3a^2}$.

The sum $3a^2 + 3ab + b^2$ is called the **Complete Divisor.**

(iv.) *If the product of the complete divisor by the second term of the root be subtracted from the first remainder, the second remainder will be* 0.

The work may be arranged as follows:

$a^3+3a^2b+3ab^2+b^3$	$a+b$	
a^3	$3a^2$	trial divisor
$3a^2b$	$3a^2b \div 3a^2 = b,$	second term of root
	$3a^2+3ab+b^2,$	complete divisor
$3a^2b+3ab^2+b^3$	$=(3a^2+3ab+b^2)\times b$	

21. Ex. 1. Find the cube root of $27x^3+54x^2y+36xy^2+8y^3$.

The work, arranged as above, is:

$27x^3+54x^2y+36xy^2+8y^3$	$3x+2y$
$27x^3$	$3(3x)^2 = 27x^2$
$54x^2y$	
$54x^2y+36xy^2+8y^3$	$(27x^2+18xy+4y^2)$

The cube root of $27x^3$ is $3x$, the first term of the root. The trial divisor is $3(3x)^2 = 27x^2$.

The second term of the root is $\dfrac{54x^2y}{27x^2}, = 2y$. The complete divisor is

$$3(3x)^2 + 3(3x)(2y) + (2y)^2, \;= 27x^2 + 18xy + 4y^2,$$

which is multiplied by the second term of the root, giving

$$54x^2y + 36xy^2 + 8y^3.$$

22. The preceding method can be extended to find cube roots which are multinomials of any number of terms, as the method of finding square roots was extended. The work consists of repetitions of the following steps:

After one or more terms of the root have been found, obtain each succeeding term by dividing the first term of the remainder at that stage by three times the square of the first term of the root.

Find the next remainder by subtracting from the last remainder the expression $(3a^2 + 3ab + b^2)b$, *wherein* a *stands for the part of the root already found, and* b *for the term last found.*

23. The given multinomial should be arranged to powers of a letter of arrangement.

Ex.

$$\begin{array}{l|l}
27-27x+90x^2-55x^3+90x^4-27x^5+27x^6 & \underline{3-x+3x^2} \\
\underline{27} & 3(3)^2=27 \\
\quad -27x & \\
\quad \underline{-27x+9x^2-x^3} & \underline{3(3)^2+3(3)(-x)+(-x)^2=27-9x+x^2} \\
\qquad\qquad 81x^2-54x^3 & 3(3-x)^2+3(3-x)(3x^2)+(3x^2)^2= \\
\qquad\qquad \underline{81x^2-54x^3+90x^4-27x^5+27x^6} & \underline{27-18x+30x^2-9x^3+9x^4}
\end{array}$$

EXERCISES III.

Find the cube root of each of the following expressions:

1. $x^3 + 9x^2 + 27x + 27$.
2. $1 - 6x + 12x^2 - 8x^3$.
3. $64a^3 + 240a^2b + 300ab^2 + 125b^3$.
4. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

5. $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1.$

6. $156a^4 - 144a^5 - 99a^3 + 64a^6 + 39a^2 - 9a + 1.$

7. $1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6.$

8. $1 - 6x + 9x^2 + 4x^3 - 9x^4 - 6x^5 - x^6.$

9. $8x^3 - 12x^2 + 12x - 7 + \frac{3}{x} - \frac{3}{4x^2} + \frac{1}{8x^3}.$

10. $27a^6x^6 + 54a^5x^5 + 9a^4x^4 - 28a^3x^3 - 3a^2x^2 + 6ax - 1.$

11. $8a^6 + 48a^5b + 60a^4b^2 - 80a^3b^3 - 90a^2b^4 + 108ab^5 - 27b^6.$

12. $x^3 + 3x^7 - 9x^{11} - 27x^{15} - 6x^5 - 54x^{13} + 28x^9.$

13. $108a^5 - 48a^4 + 8a^3 + 54a^7 - 12a^8 + a^9 - 112a^6.$

14. $8a^6 - 48a^5x + 60a^4x^2 - 27x^6 - 108ax^5 - 90a^2x^4 + 80a^3x^3.$

15. $1 + 3x - 8x^3 - 6x^4 + 6x^5 + 8x^6 - 3x^8 - x^9.$

16. $\frac{125y^6}{x^6} - \frac{150y^5}{x^5} - \frac{165y^4}{x^4} + \frac{172y^3}{x^3} + \frac{99y^2}{x^2} - \frac{54y}{x} - 27.$

ROOTS OF ARITHMETICAL NUMBERS.

Square Roots.

24. Since the squares of the numbers 1, 2, 3, ⋯, 9, 10, are 1, 4, 9, ⋯, 81, 100, respectively, the square root of an integer of *one or two* digits is a number of *one* digit.

Since the squares of the numbers 10, 11, ⋯, 100, are 100, 121, ⋯, 10000, the square root of an integer of *three or four* digits is a number of *two* digits; and so on.

Therefore, *to find the number of digits in the square root of a given integer, we first mark off the digits from right to left in groups of two. The number of digits in the square root will be equal to the number of groups, counting any one digit remaining on the left as a group.*

25. The method of finding square roots of numbers is then derived from the identity

$$(a + b)^2 = a^2 + (2a + b)b, \tag{1}$$

wherein a denotes *tens* and b denotes *units*, if the square root is a number of two digits.

26. Ex. 1. Find the square root of 1296.

We see that the root is a number of *two* digits, since the given number divides into *two* groups. The digit in the *tens'* place is 3, the square root of 9, the square next less than 12. Therefore, in the identity (1), a denotes 3 *tens*, or 30.

The work then proceeds as follows:

	$a+b$	
$12'\,96$	$30+6=36$	
$9\ 00$	$2a=60,$ trial divisor	(1)
$3\ 96$	$(2ab+b^2)\div 2a=396\div 60=6+$	(2)
$3\ 96$	$=(2a+b)\times b=(60+6)\times 6$	(3)

The first remainder, 396, is equal to $2ab+b^2$, and cannot be separated into the sum of two terms, one of which is $2ab$. We cannot, therefore, determine b by dividing $2ab$ by $2a$, as in finding square roots of algebraic expressions. Consequently step (2) *suggests* the value of b, but does not definitely determine it. As a rule, we take the integral part of the quotient, 6 in the above example, and test that value by step (3).

This method may be extended to find roots which contain any number of digits. At any stage of the work a stands for the part of the root already found, and b for the digit to be found.

Ex. 2. Find the square root of 51529.

The root is a number of *three* digits, since the given number divides into *three* groups. The digit in the *hundreds'* place is 2, the square root of 4, the square next less than 5. Therefore in the identity (1), a denotes 2 *hundreds*, or 200, in the first stage of the work. The work then proceeds as follows:

$5'\,15'\,29$	$200+20+7=227$	
$4\ 00\ 00$	$2a=400,$ trial divisor	(1)
$1\ 15\ 29$	$(2ab+b^2)\div 2a=11529\div 400=20+$	(2)
$84\ 00$	$=(2a+b)b=(400+20)\times 20$	(3)
$31\ 29$	$(2ab+b^2)\div 2a=3129\div 440=7+$	(4)
$31\ 29$	$=(2a+b)b=(440+7)\times 7$	(5)

In the second stage of the work, a stands for the part of the root already found, 220, and b for the next figure of the root. In practice the work may be arranged more compactly, omitting unnecessary ciphers, and in each remainder writing only the next group of figures. Thus:

$$\begin{array}{l|ll}
5'\,15'\,29 & \underline{227} & \\
\underline{4} & & \\
1\;15 & 11 \div 4 = 2 + & \qquad (2) \\
\underline{\;84} & \underline{42} & \\
\;31\;\;29 & 312 \div 44 = 7 + & \qquad (4) \\
\;\underline{31\;\;29} & \underline{447} &
\end{array}$$

Observe that the trial divisor at any stage is twice the part of the root already found, as in (2) and (4).

27. The abbreviated work in the last example illustrates the following method:

After one or more figures of the root have been found, obtain the next figure of the root by dividing the remainder at that stage (omitting the last figure), by the trial divisor at that stage.

See lines (2) and (4).

Annex this quotient to the part of the root already found, and also to the trial divisor to form the complete divisor.

Find the next remainder by subtracting from the last remainder the product of the complete divisor and the figure of the root last found.

28. Since the number of decimal places in the square of a decimal fraction is twice the number of decimal places in the fraction, the number of decimal places in the square root of a decimal fraction is one-half the number of decimal places in the fraction.

Consequently, in finding the square root of a decimal fraction, the decimal places are divided into groups of two from the decimal point to the right, and the integral places from the decimal point to the left as before.

Ex.

$$
\begin{array}{r|l}
14'46.28'09 & \underline{38.03} \\
9 & \\
\hline
5\;46 & \\
5\;44 & 68 \\
\hline
2.28\;09 & \\
2.28\;09 & 76.03 \\
\hline
\end{array}
$$

In finding the second figure of the root, we have $\frac{54}{6} = 9$; but $69 \times 9 = 621$, which is greater than 546, from which it is to be subtracted. Hence we take the next less figure 8.

EXERCISES IV.

Find the square root of each of the following numbers:

1. 196. **2.** 841. **3.** 1296. **4.** 65.61. **5.** 7396.

6. 3481. **7.** 667489. **8.** 170569. **9.** 1664.64.

10. 582169. **11.** 1.737124. **12.** 556.0164. **13.** .00099225.

Cube Roots.

29. Since the cubes of the numbers 1, 2, 3, ⋯, 9, 10, are 1, 8, 27, ⋯, 729, 1000, respectively, the cube root of any integer of *one, two,* or *three* digits is a number of *one* digit. *The cube roots of such numbers can be found only by inspection.*

Since the cubes of 10, 11, ⋯, 100 are 1000, 1331, ⋯, 1000000, respectively, the cube root of any integer of *four, five,* or *six* digits is a number of *two* digits, and so on.

Therefore, to find the number of digits in the cube root of a given integer, we first mark off the digits from right to left in groups of *three.* The number of digits in the cube root will be equal to the number of groups, counting one or two digits remaining on the left as a group.

30. The method of finding cube roots of numbers is derived from the identity

$$(a + b)^3 = a^3 + (3a^2 + 3ab + b^2)b, \qquad (1)$$

wherein a denotes *tens*, and b denotes *units*, if the cube root is a number of two digits.

Ex. Find the cube root of 59319.

The digit in the *tens'* place of the root is 3, the cube root of 27, the cube next less than 59. Therefore in identity (1), a denotes 3 *tens*, or 30. The work may be arranged as follows:

$$
\begin{array}{r|l r}
59'319 & \quad a+b & \\
\underline{27\,000} & \underline{30+9} & \\
32\,319 & 3a^2 = 3(30)^2 = 2700 & (1) \\
 & (3a^2b + 3ab^2 + b^3) \div 3a^2 = 32319 + 2700 = 9 + & (2) \\
 & \qquad 3a^2 = 3(30)^2 \ = 2700 & \\
 & \qquad 3ab = 3(30)9 = \ \ 810 & \\
 & \qquad\ \ b^2 = \qquad 9^2 = \ \ \underline{\ \ 81} & \\
\underline{32\,319} & \underline{= (3a^2 + 3ab + b^2) \times b = 3591 \times 9} & (3)
\end{array}
$$

As in finding square roots of numbers, step (2) *suggests* the value of b, but does not definitely determine it. If the value of b makes $(3a^2 + 3ab + b^2) \times b$ greater than the number from which it is to be subtracted, we must try the next less number.

In practice the work may be arranged more compactly, omitting unnecessary ciphers, and in each remainder writing only the next group of figures; thus

$$
\begin{array}{r|r r}
59'\,319 & \underline{39} & \\
\underline{27\qquad} & & \\
32\ 319 & 2700 & (1) \\
 & 810 & (2) \\
 & \underline{81} & (3) \\
\underline{32\ 319} & \underline{3591} &
\end{array}
$$

31. The preceding method may be extended to find roots that contain any number of digits.

At any stage of the work a stands for the part of the root already found, and b for the digit to be found.

The method consists of repetitions of the following steps:

The trial divisor at any stage is three times the square of the part of the root already found; as 27 in the preceding example.

After one or more figures of the root have been found, obtain the next figure of the root by dividing the remainder at that stage (omitting the last two figures) by the trial divisor. In the last example, $9+ = 323 \div 27$.

Annex this quotient to the part of the root already found.

To obtain the complete divisor, add to the trial divisor (with two ciphers annexed) three times the product of the part of the root already found (with one cipher annexed) by the figure of the root just found, and also the square of the figure of the root just found.

Find the next remainder by subtracting from the last remainder the product of the complete divisor and the figure of the root last found.

32. Evidently, in finding the cube root of a decimal fraction the decimal places are divided into groups of *three* figures from the decimal point to the right, and the integral places from the decimal point to the left as before.

EXERCISES V.

Find the cube root of each of the following numbers:

1. 2744. **2.** 39304. **3.** 110.592. **4.** 328509.

5. 1.191016. **6.** 74088000. **7.** 340068392. **8.** 426.957777.

9. 584067.412279. **10.** 375601280.458951. **11.** .041063625.

HIGHER ROOTS.

33. Since $\sqrt[4]{a^4} = a$, and $\sqrt{\sqrt{a^4}} = \sqrt{a^2} = a$,

therefore, $\sqrt[4]{a^4} = \sqrt{\sqrt{a^4}}$.

Since $\sqrt[6]{a^6} = a$, and $\sqrt[3]{\sqrt{a^6}} = \sqrt[3]{a^3} = a$,

therefore, $\sqrt[6]{a^6} = \sqrt[3]{\sqrt{a^6}}$.

In general, since

$$\sqrt[pq]{a^{pq}} = a, \text{ and } \sqrt[p]{\sqrt[q]{a^{pq}}} = \sqrt[p]{a^p} = a,$$

therefore, $\sqrt[pq]{a^{pq}} = \sqrt[p]{\sqrt[q]{a^{pq}}}$.

That is, *the pqth root of a number is the pth root of the qth root of the number.*

In particular, the fourth root is the square root of the square root, the sixth root is the cube root of the square root.

EXERCISES VI.

Find the fourth root of each of the following expressions:

1. $x^8 + 4x^6 + 6x^4 + 4x^2 + 1.$

2. $a^8 + 4a^7b + 10a^6b^2 + 16a^5b^3 + 19a^4b^4 + 16a^3b^5 + 10a^2b^6 + 4ab^7 + b^8.$

3. $16x^8 - 160x^7 + 408x^6 + 440x^5 - 2111x^4 - 1320x^3 + 3672x^2 + 4320x + 1296.$

4. $625x^8 + 5500x^7 + 17150x^6 + 20020x^5 + 721x^4 - 8008x^3 + 2744x^2 - 352x + 16.$

Find the sixth roots of each of the following expressions:

5. $64x^{12} - 192x^{10} + 240x^8 - 160x^6 + 60x^4 - 12x^2 + 1.$

6. $a^{12} + 6a^{11}b + 21a^{10}b^2 + 50a^9b^3 + 90a^8b^4 + 126a^7b^5 + 141a^6b^6 + 126a^5b^7 + 90a^4b^8 + 50a^3b^9 + 21a^2b^{10} + 6ab^{11} + b^{12}.$

Find the value of each of the following indicated roots:

7. $\sqrt[4]{279841}.$ 8. $\sqrt[6]{3010936384}.$ 9. $\sqrt[4]{164204746.7776}.$

CHAPTER XV.

SURDS.

1. In Ch. XIV. we considered only roots of powers whose exponents were multiples of the indices of the required roots. Such roots as $\sqrt{2}$, $\sqrt[3]{a^2}$, etc., were excluded.

2. Such roots as $\sqrt{2}$, $\sqrt[3]{a^2}$, etc., cannot be expressed either as integers or as fractions. Thus, there is no integer or fraction whose square is 2.

But it is there proved that the value of such a root can be found approximately to any degree of accuracy.

E.g., approximate values of $\sqrt{2}$ are 1.4, 1.41, 1.414, etc.

3. It is also proved that these roots obey the fundamental laws of Algebra; as $\sqrt{2} \times \sqrt{3} = \sqrt{3} \times \sqrt{2}$, etc.

4. An **Irrational Number** is a number which cannot be expressed as an integer or as a fraction; as $\sqrt{2}$, $\sqrt[3]{a^2}$.

An **Irrational Expression** is an expression which involves an irrational number; as $\sqrt[3]{5}$, $a + \sqrt{b}$.

5. A **Rational Number** is a number which can be expressed as an integer or as a fraction; as $2, \dfrac{2x}{3y}, \sqrt[3]{(27a^6)}$.

A **Rational Expression** is an expression which involves only rational numbers; as $\frac{2}{3}a + \frac{1}{2}b$, $ab + \sqrt{a^2}$.

6. A **Radical** is an indicated root of a number or expression; as $\sqrt{7}$, $\sqrt{9}$, $\sqrt[3]{(a+b)}$.

A **Radical Expression** is an expression which contains radicals; as $2\sqrt{7}$, $\sqrt{x} + \sqrt{y}$, $\sqrt{(a + \sqrt{b})}$.

7. A **Surd** is an irrational root of a rational number; as $\sqrt{7}$, $\sqrt{a}$.

Observe that $\sqrt{(1+\sqrt{7})}$ is not a surd, since $1+\sqrt{7}$ is not a rational number.

8. The **Order** of a surd is indicated by the index. Thus, $\sqrt{a}$ is a surd of the *second order*, or a *quadratic surd;* $\sqrt[3]{5}$ is a surd of the *third order;* and so on.

Principles of Surds.

9. As in Ch. XIV., we limit the radicands to positive values, and the roots to positive roots.

10. The principles established in Ch. XIV., Art. 13, and their proofs, hold also for surds. For, any positive number is a power of either a rational or an irrational number.

Thus, $4=2^2,\ 3=(\sqrt{3})^2,\ a=(\sqrt[q]{a})^q.$

We have $\sqrt{(ab)}=\sqrt{[(\sqrt{a})^2(\sqrt{b})^2]}=\sqrt{a}\times\sqrt{b}$; and so on.

Therefore,

(i.) $\sqrt[q]{a^{nq}}=a^{\frac{nq}{q}}=a^n.$ [Ch. XIV., Art. 13, (i.).]

(ii.) $\sqrt[q]{(ab)}=\sqrt[q]{a}\times\sqrt[q]{b}.$ [Ch. XIV., Art. 13, (ii.).]

(iii.) $\sqrt[q]{\frac{a}{b}}=\frac{\sqrt[q]{a}}{\sqrt[q]{b}}.$ [Ch. XIV., Art. 13, (iii.).]

Reduction of Surds.

11. A surd is in its *simplest form* when the radicand is integral, and does not contain a factor with an exponent equal to or a multiple of the index of the root; as $\sqrt{2}$, $\sqrt[3]{(a^2b)}$, $\sqrt[n]{a^m}$.

12. Ex. 1. $\sqrt{80}=\sqrt{(16\times 5)}=\sqrt{16}\times\sqrt{5}=4\sqrt{5}.$

Ex. 2. $\sqrt{(18\,a^5b^2)}=\sqrt{(9\,a^4b^2\times 2\,a)}=\sqrt{(9\,a^4b^2)}\times\sqrt{(2\,a)}$
$=3\,a^2b\sqrt{(2\,a)}.$

These examples illustrate the following method of reducing a surd to its simplest form:

Separate the radicand into two factors, one a product of powers with the highest exponents which are multiples of the given index. Multiply the rational root of this factor by the irrational root of the second factor.

Ex. 3. $\sqrt[3]{(48\,x^5y^3)} = \sqrt[3]{(8\,x^3y^3 \times 6\,x^2)} = 2\,xy\sqrt[3]{(6\,x^2)}$.

Ex. 4. $\sqrt[n]{(a^{n+1}b^{2n+2})} = \sqrt[n]{(a^nb^{2n} \times ab^2)} = ab^2\sqrt[n]{(ab^2)}$.

EXERCISES I.

Reduce each of the following surds to its simplest form:

1. $\sqrt{32}$. 2. $\sqrt{75}$. 3. $\sqrt{108}$. 4. $\sqrt{x^3}$.
5. $\sqrt{(a^2b)}$. 6. $\sqrt{(a^4b^5)}$. 7. $\sqrt{(8\,a^7x^{11})}$. 8. $\sqrt{(50\,ax^2y^3)}$.
9. $\sqrt[3]{192}$. 10. $\sqrt[3]{-10\frac{1}{8}}$. 11. $\sqrt[3]{-a^{10}}$. 12. $\sqrt[3]{(24\,b^3c^4)}$.
13. $\sqrt[3]{(16\,a^5x^9)}$. 14. $\sqrt[4]{(32\,a^6x^8)}$. 15. $\sqrt[5]{(-96\,x^5y^{12})}$.
16. $\sqrt[n]{x^{3n+4}}$. 17. $\sqrt[n+1]{a^{2n+3}}$. 18. $\sqrt[n-1]{a^{2n+1}}$.
19. $\sqrt{(a^{2n}b^{2n+1})}$. 20. $\sqrt[3]{(-x^{7n}b^{3n})}$. 21. $\sqrt[n]{(a^{2n+1}b)}$.
22. $\sqrt{(a^2b^2 + a^2c^2)}$. 23. $\sqrt{(ab^3c^4 - b^2c^6)}$.
24. $\sqrt{(b-c)(b^3-c^3)}$. 25. $\sqrt{(a^2-1)(1+a)}$.
26. $\sqrt{(9\,x^3 - 18\,x^2 + 9\,x)}$. 27. $\sqrt{(4\,a^3b - 8\,a^2b^2 + 4\,ab^3)}$.

13. When the Expression under the Radical Sign is a Fraction. — In this case we reduce the numerator and denominator separately by Art. 10 (iii.).

Ex. 1. $\sqrt{\dfrac{3\,a^2}{4\,b^2}} = \dfrac{\sqrt{(3\,a^2)}}{\sqrt{(4\,b^2)}} = \dfrac{a\sqrt{3}}{2\,b}$.

Ex. 2. $\sqrt{\dfrac{8\,x^2}{y}} = \sqrt{\dfrac{8\,x^2y}{y^2}} = \dfrac{\sqrt{(8\,x^2y)}}{\sqrt{y^2}} = \dfrac{2\,x\sqrt{(2\,y)}}{y}$.

When the required root of the denominator is not rational, we proceed as in Ex. 2:

First multiply both terms of the fraction by the expression of lowest degree which will make the denominator a power with an exponent equal to the index of the root. Then proceed as before.

Ex. 3. $\sqrt[3]{\dfrac{7}{12}} = \sqrt[3]{\dfrac{7}{4 \times 3}} = \sqrt[3]{\dfrac{7 \times 2 \times 9}{8 \times 27}} = \dfrac{\sqrt[3]{126}}{2 \times 3} = \frac{1}{6}\sqrt[3]{126}$.

EXERCISES II.

Simplify each of the following expressions:

1. $\sqrt{\dfrac{5}{9}}$. 2. $\sqrt{\dfrac{a^3}{4}}$. 3. $\sqrt[3]{\dfrac{32\,x^3y^4}{27\,a^6}}$. 4. $\sqrt[4]{\dfrac{81\,a^5b^8}{16\,x^4y^5}}$.

5. $\sqrt{\frac{1}{5}}$.	**6.** $2\sqrt{\frac{1}{2}}$.	**7.** $\sqrt{\frac{1}{8}}$.	**8.** $6\sqrt{\frac{2}{3}}$.
9. $\sqrt[3]{\frac{1}{2}}$.	**10.** $\sqrt[3]{\frac{8}{9}}$.	**11.** $6\sqrt[3]{\frac{2}{3}}$.	**12.** $8\sqrt[3]{\frac{3}{4}}$.
13. $\sqrt[4]{\frac{3}{8}}$.	**14.** $\sqrt[4]{\frac{5}{9}}$.	**15.** $\sqrt[5]{\frac{2}{3}}$.	**16.** $\sqrt[5]{\frac{2}{343}}$.
17. $\sqrt{\frac{64\,a}{81\,b}}$.	**18.** $\sqrt{\frac{18\,a^2x^3}{125\,b^5}}$.	**19.** $\sqrt{\frac{16\,a^8}{45\,b^3x^5}}$.	**20.** $\sqrt[3]{\frac{x^3}{y}}$.
21. $\sqrt[3]{\frac{a}{b^2}}$.	**22.** $\sqrt[3]{\frac{a}{27\,b}}$.	**23.** $\sqrt[3]{\frac{3\,a^2x^3}{4\,b^3y^4}}$.	**24.** $\sqrt[3]{\frac{ax^{4n}}{8\,b^2}}$.
25. $\sqrt[3]{\frac{128\,a^7x^3}{b^6y^{13}}}$.	**26.** $\sqrt[4]{\frac{16\,a^5x^{16}}{b^3y^{11}}}$.	**27.** $\sqrt[5]{\frac{a^6b^8}{x^8}}$.	**28.** $\sqrt[6]{\frac{a^6}{6\,b^7x^{23n}}}$.

14. When the index of the root and the exponent of the radicand have a common factor. We have

$$(\sqrt[3]{a^2})^{12} = [(\sqrt[3]{a^2})^3]^4 = [a^2]^4 = a^8.$$

Therefore, $$\sqrt[12]{a^8} = \sqrt[3]{a^2} = \sqrt[\frac{12}{4}]{a^{\frac{8}{4}}}.$$

This example illustrates the following method:

Divide the index of the root and the exponent of the radicand by their H. C. F.

In general, $$\sqrt[nq]{a^{np}} = \sqrt[\frac{nq}{n}]{a^{\frac{np}{n}}} = \sqrt[q]{a^p}.$$

For, $$(\sqrt[q]{a^p})^{nq} = [(\sqrt[q]{a^p})^q]^n = [a^p]^n = a^{np}.$$

Therefore, $$\sqrt[nq]{a^{np}} = \sqrt[q]{a^p}.$$

Ex. 1. $\sqrt[4]{a^2} = \sqrt{a}$.

Ex. 2. $\sqrt[6]{9} = \sqrt[6]{3^2} = \sqrt[3]{3}$.

Ex. 3. $\sqrt[6]{(27\,a^3b^6)} = \sqrt[6]{b^6} \times \sqrt[6]{(3\,a)^3} = b\sqrt{(3\,a)}$.

EXERCISES III.

Simplify each of the following expressions:

1. $\sqrt[4]{25}$.	**2.** $\sqrt[4]{49}$.	**3.** $\sqrt[6]{8}$.	**4.** $\sqrt[6]{25}$.
5. $\sqrt[8]{16}$.	**6.** $\sqrt[8]{81}$.	**7.** $\sqrt[4]{(81\,a^2)}$.	**8.** $\sqrt[6]{(27\,a^3)}$.

9. $\sqrt[4]{(4\,a^4x^2)}$.	**10.** $\sqrt[6]{(125\,a^3x^6)}$.	**11.** $\sqrt[8]{(49\,a^4x^2)}$.
12. $\sqrt[6]{(8\,a^9b^{15})}$.	**13.** $\sqrt[12]{(64\,a^8x^{10})}$.	**14.** $\sqrt[15n]{(a^{30n}b^{20n})}$.
15. $\sqrt[4]{\frac{25}{49}}$.	**16.** $\sqrt[10]{\frac{32}{x^{15}y^{20}}}$.	**17.** $\sqrt[mx]{\frac{1}{a^{nx}}}$.

Addition and Subtraction of Surds.

15. **Similar** or **Like Surds** are rational multiples of one and the same simple monomial surd; as $\sqrt{12}$, $= 2\sqrt{3}$, and $5\sqrt{3}$.

The rational factor is called the coefficient of the surd factor.

16. *Like* surds, or such surds as can be reduced to *like* surds, can be united by algebraic addition into a single like surd.

Ex. **1.** $\sqrt{12}+2\sqrt{27}-9\sqrt{48}=2\sqrt{3}+6\sqrt{3}-36\sqrt{3}=-28\sqrt{3}.$

Ex. **2.** $8\sqrt[3]{40}+3\sqrt[3]{135}-2\sqrt[3]{625}=16\sqrt[3]{5}+9\sqrt[3]{5}-10\sqrt[3]{5}$
$$=15\sqrt[3]{5}.$$

Ex. **3.** $\sqrt{2}-\sqrt{\tfrac{1}{2}}+\sqrt{.02}=\sqrt{2}-\tfrac{1}{2}\sqrt{2}+\tfrac{1}{10}\sqrt{2}=\tfrac{3}{5}\sqrt{2}.$

Ex. **4.** $\sqrt{(a^5b)}+2\sqrt{(a^3b^3)}+\sqrt{(ab^5)}$
$$=a^2\sqrt{(ab)}+2\,ab\sqrt{(ab)}+b^2\sqrt{(ab)}=(a+b)^2\sqrt{(ab)}.$$

These examples illustrate the method: *Reduce each surd to its simplest form, and take the algebraic sum of the coefficients.*

EXERCISES IV.

Simplify:

1. $5\sqrt{2}+3\sqrt{2}-7\sqrt{2}.$ **2.** $3\sqrt{a}-5\sqrt{a}+7\sqrt{a}.$

3. $8\sqrt[3]{9}-3\sqrt[3]{9}+7\sqrt[3]{9}.$ **4.** $2\sqrt[4]{x}-5\sqrt[4]{x}-9\sqrt[4]{x}.$

5. $\sqrt{5}+\sqrt{20}.$ **6.** $\sqrt{90}-5\sqrt{40}.$ **7.** $\sqrt{(16\,a)}-3\sqrt{a}.$

8. $8\sqrt{(9\,b)}-3\sqrt{(16\,b)}.$ **9.** $\sqrt[3]{16}-3\sqrt[3]{54}.$ **10.** $2\sqrt[3]{81}-5\sqrt[3]{24}.$

11. $2\sqrt[3]{(8\,x)}+5\sqrt[3]{(27\,x)}.$ **12.** $6\sqrt[3]{(108\,a)}-3\sqrt[3]{(500\,a)}.$

13. $x\sqrt{(xy^3)}+y\sqrt{(x^3y)}.$ **14.** $5\,a\sqrt{(3\,b^2)}-b\sqrt{(48\,a^2)}.$

15. $\sqrt{2}+3\sqrt{8}-\sqrt{50}.$ **16.** $3\sqrt{3}+\sqrt{27}-11\sqrt{48}.$

17. $3\sqrt{6}+2\sqrt{24}-\sqrt{54}.$ **18.** $30\sqrt{20}+4\sqrt{45}-11\sqrt{245}.$

19. $3\sqrt{75}+4\tfrac{1}{2}\sqrt{192}-2\tfrac{3}{4}\sqrt{12}.$ **20.** $\sqrt{2\tfrac{9}{20}}+\sqrt{5\tfrac{41}{80}}-\sqrt{\tfrac{9}{5}}.$

21. $4\sqrt{\tfrac{3}{4}}-\tfrac{2}{7}\sqrt{\tfrac{3}{16}}-2\sqrt{27}.$ **22.** $2\sqrt{\tfrac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\tfrac{3}{5}}.$

23. $8\sqrt[3]{48}+3\sqrt[3]{162}-2\sqrt[3]{384}.$ **24.** $5\sqrt[3]{54}+9\sqrt[3]{250}-\sqrt[3]{686}.$

25. $2\tfrac{3}{5}\sqrt[3]{500}+\tfrac{3}{4}\sqrt[3]{256}-3\tfrac{1}{2}\sqrt[3]{32}-\tfrac{2}{3}\sqrt[3]{108}.$

26. $\sqrt[3]{40}-5\sqrt[3]{\tfrac{1}{25}}+4\sqrt[3]{(-.625)}-\tfrac{2}{3}\sqrt[3]{16\tfrac{7}{8}}.$

27. $2\sqrt{3}-\sqrt{12}+\sqrt[4]{9}$. **28.** $\sqrt[3]{24}+3\sqrt[6]{9}-5\sqrt[3]{192}$.

29. $\sqrt{(4\,a^3)}+\sqrt{(9\,a^3)}+\sqrt{(25\,a^3)}-\sqrt{(81\,a^3)}$.

30. $\sqrt{(12\,a^2b)}+\sqrt{(75\,a^2b)}-\sqrt{(27\,a^2b)}$.

31. $\sqrt[3]{(64\,a^8b^5)}+\sqrt[3]{(125\,a^8b^5)}-\sqrt[3]{(a^8b^5)}$.

32. $a\sqrt{(a^3b^7)}+b^2\sqrt{(a^5b^3)}-2\,ab^2\sqrt{(a^3b^3)}+\sqrt[8]{(a^{20}b^{28})}$.

33. $\sqrt[4]{(9\,a^6b^2)}+\sqrt{(27\,a^3b)}+5\sqrt[4]{(729\,a^6b^2)}$.

34. $\sqrt{(9\,a+27)}+3\sqrt{(4\,a+12)}$.

35. $\sqrt{(4\,a^3+4\,a^2b)}+\sqrt{(4\,ab^2+4\,b^3)}$.

36. $7\,x\sqrt{(25\,a+75)}-5\sqrt{(9\,x^2a+27\,x^2)}$.

37. $2\sqrt{(2\,x^3)}-\sqrt{(8\,x)}-\sqrt{(2\,x^3-4\,x^2+2\,x)}$.

Reduction of Surds of Different Orders to Equivalent Surds of the Same Order.

17. The converse of the principle of Art. 14 evidently holds. That is,

$$\sqrt[q]{a^q}=\sqrt[nq]{a^{nq}}.$$

Ex. **1.** Reduce $\sqrt{2}$, $\sqrt[4]{(3\,a)}$, and $\sqrt[6]{(5\,b)}$ to equivalent surds of the same order.

We have
$$\sqrt{2}=\sqrt[12]{2^6}\quad=\sqrt[12]{64};$$
$$\sqrt[4]{(3\,a)}=\sqrt[12]{(3\,a)^3}=\sqrt[12]{(27\,a^3)};$$
$$\sqrt[6]{(5\,b)}=\sqrt[12]{(5\,b)^2}=\sqrt[12]{(25\,b^2)}.$$

We thus have the following method:

Take the L. C. M. of the given indices as the common index of the equivalent surds. Raise each radicand to a power whose exponent is equal to the quotient obtained by dividing this L. C. M. by the given index.

18. Any rational number can be expressed in the form of a surd.

Ex. **2.** $2=\sqrt{4}=\sqrt[3]{8}=\cdots;\ a=\sqrt{a^2}=\sqrt[3]{a^3}=\sqrt[q]{a^q}$.

Write under the radical sign a power of the number whose exponent is equal to the index.

19. Two surds, or a surd and a rational number, can be compared by first reducing them to equivalent surds of the same order.

Ex. **3.** Which is greater, $\sqrt{2}$ or $\sqrt[3]{3}$?

We have $\sqrt{2} = \sqrt[6]{8}$, and $\sqrt[3]{3} = \sqrt[6]{9}$.

Since $9 > 8$, therefore $\sqrt[6]{9} > \sqrt[6]{8}$, or $\sqrt[3]{3} > \sqrt{2}$.

EXERCISES V.

Reduce to equivalent surds of the same order:

1. $\sqrt{2}$, $\sqrt[4]{5}$. **2.** $\sqrt{3}$, $\sqrt[4]{6}$. **3.** $\sqrt{7}$, $\sqrt[3]{10}$.

4. $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{4}}$. **5.** 5, $\sqrt[4]{10}$. **6.** 6, $\sqrt[3]{4}$.

7. $\sqrt[6]{2}$, $\sqrt[8]{3}$. **8.** $\sqrt[10]{15}$, $\sqrt[15]{10}$. **9.** $\sqrt[n]{a^3}$, $\sqrt[m]{b^5}$.

10. $\sqrt{5}$, $\sqrt[3]{10}$, $\sqrt[4]{15}$. **11.** $\sqrt[4]{2}$, $\sqrt{3}$, $\sqrt[8]{5}$. **12.** $\sqrt[2m]{a^3}$, $\sqrt[4m]{b^3}$, $\sqrt[6m]{c^5}$.

Which is the greater,

13. $2\sqrt{3}$ or $3\sqrt{2}$? **14.** $\sqrt{5}$ or $\sqrt[3]{10}$? **15.** $\frac{1}{2}\sqrt[3]{25}$ or $\frac{1}{3}\sqrt{11}$?

16. $\sqrt[3]{a^2}$ or $\sqrt{a}$, when $a < 1$? **17.** $\sqrt[4]{x^3}$ or $\sqrt[5]{x^4}$, when $x > 1$?

Which is the greatest,

18. $\sqrt{3}$, $\sqrt[3]{5}$, or $\sqrt[4]{10}$? **19.** $\sqrt{\frac{2}{3}}$, $\sqrt[3]{\frac{3}{2}}$, or $\sqrt[4]{\frac{7}{4}}$?

Multiplication of Surds.

20. Multiplication of Monomial Surds. — The converse of the principle of Art. 10 (ii.) evidently holds. That is,

$$\sqrt[q]{a} \times \sqrt[q]{b} = \sqrt[q]{(ab)}.$$

Ex. **1.** $5\sqrt[3]{4} \times 2\sqrt[3]{6} = 10\sqrt[3]{24} = 20\sqrt[3]{3}$.

Ex. **2.** $\sqrt{a} \times \sqrt[3]{a^2} = \sqrt[6]{a^3} \times \sqrt[6]{a^4} = \sqrt[6]{a^7} = a\sqrt[6]{a}$.

We thus have the following method:

Reduce surds of different orders to equivalent surds of the same order.

Multiply the product of the coefficients by the product of the surd factors.

Simplify the result.

Ex. 3. $\sqrt{12} \times \sqrt[3]{36} = \sqrt{(4 \times 3)} \times \sqrt[3]{(4 \times 9)} = 2\sqrt{3} \times \sqrt[3]{(2^2 \times 3^2)}$
$= 2\sqrt[6]{3^3} \times \sqrt[6]{(2^4 \times 3^4)} = 2\sqrt[6]{(2^4 \times 3^7)}$
$= 6\sqrt[6]{(2^4 \times 3)} = 6\sqrt[6]{48}.$

When the radicands contain numerical factors it is advisable to express them as powers of the smallest possible bases.

21. It is frequently desirable to introduce the coefficient of a surd under the radical sign.

Ex. 4. $4\sqrt{5} = \sqrt{16} \times \sqrt{5} = \sqrt{80}.$

Ex. 5. $3a\sqrt[3]{(2ab)} = \sqrt[3]{(27a^3)} \times \sqrt[3]{(2ab)} = \sqrt[3]{(54a^4b)}.$

EXERCISES VI.

Multiply:

1. $\sqrt{3} \times \sqrt{5}.$ **2.** $\sqrt{(5a)} \times \sqrt{(6b)}.$ **3.** $\sqrt{2} \times 2\sqrt{8}.$

4. $4\sqrt{15} \times \sqrt{45}.$ **5.** $\sqrt{\frac{5}{7}} \times \sqrt{\frac{7}{125}}.$ **6.** $2\sqrt[3]{\frac{25}{16}} \times \frac{1}{5}\sqrt[3]{\frac{2}{5}}.$

7. $3\sqrt[3]{45} \times 5\sqrt[3]{150}.$ **8.** $9\sqrt[4]{54} \times 3\sqrt[4]{24}.$

9. $\sqrt[3]{6} \times 3\sqrt[3]{36}.$ **10.** $\sqrt{a} \times \sqrt{(2a)}.$ **11.** $5\sqrt{m} \times \sqrt{mn}.$

12. $7\sqrt{(6x)} \times 4\sqrt{(18x)}.$ **13.** $\sqrt[3]{(a^2x)} \times \sqrt[3]{a}.$

14. $\sqrt[3]{(5x^2)} \times \sqrt[3]{(25xy)}.$ **15.** $\sqrt[3]{(4a^2b)} \times \sqrt[3]{(6ab^2)}.$

16. $\sqrt{(1+x)} \times \sqrt{(ax+a)}.$ **17.** $\sqrt[3]{(1-x)^2} \times \sqrt[3]{(1-x^2)}.$

18. $\sqrt{6} \times \sqrt[3]{4}.$ **19.** $\sqrt[3]{50} \times \sqrt[6]{75}.$ **20.** $\sqrt{21} \times \sqrt[4]{27}.$

21. $\sqrt[3]{20} \times \sqrt{2}.$ **22.** $\sqrt[4]{72} \times \sqrt[6]{108}.$ **23.** $\sqrt[3]{\frac{2}{3}} \times \sqrt{3}.$

24. $\sqrt{\frac{a}{b}} \times \sqrt[4]{\frac{b^3}{a}}.$ **25.** $\sqrt[3]{\frac{a^5}{b^2}} \times \sqrt[9]{\frac{b^8}{a^4}}.$ **26.** $\sqrt[6]{\frac{ax}{by}} \times \sqrt[10]{\frac{ay}{bx}}.$

27. $\sqrt[5]{54} \times 3\sqrt{6} \times 5\sqrt[3]{2}.$ **28.** $\sqrt{10} \times \sqrt[3]{100} \times \sqrt[4]{500}.$

29. $\sqrt[3]{12} \times \sqrt[4]{108} \times \sqrt[6]{486}.$ **30.** $12\sqrt[4]{14} \times \sqrt{2\frac{1}{7}} \times \sqrt[8]{\frac{49}{800}}.$

31. $\sqrt[3]{12} \times \sqrt[4]{216} \times \sqrt[6]{96}.$

32. $\sqrt{(40x)} \times \sqrt[5]{(250x)} \times \sqrt[10]{(80x^3)}.$

In each of the following expressions introduce the coefficient under the radical sign:

33. $3\sqrt{2}$. **34.** $5\sqrt{3}$. **35.** $2\sqrt[3]{25}$. **36.** $10\sqrt[3]{7}$.

37. $5\sqrt[4]{3}$. **38.** $\frac{1}{2}\sqrt{2}$. **39.** $\frac{1}{2}\sqrt[3]{4}$. **40.** $\frac{3}{4}\sqrt[3]{\frac{16}{9}}$.

41. $2a\sqrt{a}$. **42.** $5x^2\sqrt{(3xy)}$. **43.** $4a^2b\sqrt[3]{(2a)}$.

44. $a\sqrt[n]{a}$. **45.** $a^2b\sqrt[n-1]{(ab)}$. **46.** $a^{n+1}\sqrt{a^{n-2}}$.

47. $(a+b)\sqrt{\dfrac{ab}{a^2+2ab+b^2}}$. **48.** $(m-n)\sqrt{\dfrac{m+n}{m-n}}$.

22. Involution of Monomial Surds. — We have

$$(\sqrt{a})^3 = \sqrt{a} \times \sqrt{a} \times \sqrt{a} = \sqrt{(aaa)} = \sqrt{a^3}.$$

In general, $(\sqrt[q]{a})^n = \sqrt[q]{a} \times \sqrt[q]{a} \times \sqrt[q]{a} \cdots$ to n factors

$= \sqrt[q]{(aaa \cdots \text{ to } n \text{ factors})}$

or, $$(\sqrt[q]{a})^n = \sqrt[q]{a^n}.$$

That is, to raise a surd to any required power:

Raise the radicand to the required power.

Ex. 6. $(\sqrt[3]{2})^4 = \sqrt[3]{2^4} = 2\sqrt[3]{2}$.

23. If the index of the root and exponent of the required power have a common factor, the work is simplified by Art. 14:

$$(\sqrt[nq]{a})^{np}, = \sqrt[nq]{a^{np}} = \sqrt[q]{a^p}.$$

Ex. 1. $(\sqrt[6]{5})^2 = \sqrt[3]{5}$. Ex. 2. $[\sqrt[9]{(ab)}]^6 = [\sqrt[3]{(ab)}]^2 = \sqrt[3]{(a^2b^2)}$.

Ex. 3. $[5x\sqrt[8]{(32y^4)}]^2 = 25x^2\sqrt[4]{(2^5y^4)} = 50x^2y\sqrt[4]{2}$.

EXERCISES VII.

Simplify:

1. $(\sqrt{5})^2$. **2.** $(\sqrt[3]{a})^3$. **3.** $(\sqrt[4]{xy})^2$. **4.** $(\sqrt[6]{a^2b^2})^3$.

5. $(\sqrt[3]{2x})^6$. **6.** $(\sqrt{3x})^3$. **7.** $(\sqrt[3]{5a})^2$. **8.** $(3\sqrt{ax})^4$.

9. $(2\sqrt[4]{x^3})^2$. **10.** $(\sqrt[4]{a^3x^2})^2$. **11.** $(3\sqrt[4]{2})^5$. **12.** $(\frac{1}{2}\sqrt{6ab})^3$.

13. $(\sqrt[5]{a^4b})^2$. **14.** $(\sqrt[6]{8x^3y})^3$. **15.** $(\sqrt[4]{7a})^3$. **16.** $(2a\sqrt{3b})^8$.

24. Multiplication of Multinomial Surd Numbers. — The work may be arranged as in multiplication of rational multinomials.

Ex. Multiply $2\sqrt{5}+3\sqrt{2}$ by $\sqrt{5}-4\sqrt{2}$.

We have

$$\begin{array}{r}2\sqrt{5}+3\sqrt{2}\\ \underline{\sqrt{5}-4\sqrt{2}}\\ 10+3\sqrt{10}\\ \underline{-8\sqrt{10}-24}\\ 10-5\sqrt{10}-24\end{array} = -14-5\sqrt{10}.$$

25. Conjugate Surds. — Two binomial quadratic surds which differ only in the sign of a surd term are called **Conjugate Surds.**

E.g., $\sqrt{3}+\sqrt{2}$ and $-\sqrt{3}+\sqrt{2}$; $1-\sqrt{5}$ and $1+\sqrt{5}$.

Either of two conjugate surds is the conjugate of the other.

The product of two conjugate surds is a rational number.

For, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b$.

26. Type-Forms. — Many products are more easily obtained by using the type-forms given in Ch. V.

Ex.
$$\begin{aligned}(\sqrt{2}+\sqrt{3})^2&=(\sqrt{2})^2+2\sqrt{2}\times\sqrt{3}+(\sqrt{3})^2\\&=2+2\sqrt{6}+3=5+2\sqrt{6}.\end{aligned}$$

EXERCISES VIII.

Simplify each of the following expressions:

1. $(\sqrt{3}+3\sqrt{6}-5\sqrt{8})\times\sqrt{6}$.
2. $(\sqrt[3]{9}-2\sqrt[3]{45}+5\sqrt[3]{54})\times\sqrt[3]{3}$.
3. $(5+\sqrt{3})(1-3\sqrt{3})$.
4. $(\sqrt{10}-2)(\sqrt{10}+5)$.
5. $(2\sqrt{7}-5\sqrt{13})(\sqrt{91}-5)$.
6. $(\sqrt{6}+11\sqrt{5})(\sqrt{2}+4\sqrt{15})$.
7. $(x+2\sqrt{a})(x-3\sqrt{a})$
8. $(\sqrt{a}+\sqrt{b})(a\sqrt{b}-b\sqrt{a})$.
9. $(4\sqrt{3}-3\sqrt{6}+5\sqrt{2})(5\sqrt{3}-6\sqrt{2})$.
10. $(\sqrt{3}+8\sqrt{6}-7)(\sqrt{6}-2\sqrt{3}+7\sqrt{2})$.

11. $(3\sqrt{2} - 6\sqrt{5} + 2\sqrt{10})(\sqrt{2} + 3\sqrt{5} + 4\sqrt{10})$.

12. $(\sqrt{7} + \sqrt[4]{21} + \sqrt{3})(\sqrt[4]{7} - \sqrt[4]{3})$.

13. $\sqrt{(3 - 2\sqrt{2})} \times \sqrt[4]{(17 + 12\sqrt{2})}$.

14. $\sqrt[3]{(2 - \sqrt{3})} \times \sqrt{(2 + \sqrt{3})}$.

15. $(5\sqrt[3]{9} + 3\sqrt[3]{25})(\sqrt[3]{3} - \sqrt[3]{5})$.

16. $(\sqrt[4]{27} - \sqrt[4]{2})(2\sqrt[4]{3} + 3\sqrt[4]{8})$.

Find the value of each of the following expressions, without performing the actual multiplications:

17. $(\sqrt{5} - \sqrt{10})^2$. 18. $(\sqrt{6} - 4\sqrt[4]{40})^2$. 19. $(\sqrt{3} - \sqrt{6})^3$.

20. $(\sqrt{6} - 2\sqrt[3]{2})^3$. 21. $(1 + \sqrt{2} - \sqrt{3})^2$. 22. $(\sqrt{2} + \sqrt{3} + 1)^3$.

23. $(8 + 3\sqrt{7})(8 - 3\sqrt{7})$.

24. $(2\sqrt{5} - 4\sqrt{3})(2\sqrt{5} + 4\sqrt{3})$.

25. $\sqrt{(6 + \sqrt{11})} \times \sqrt{(6 - \sqrt{11})}$.

26. $\sqrt[3]{(2\sqrt{2} - 3)} \times \sqrt[3]{(2\sqrt{2} + 3)}$.

27. $[\sqrt{(7 + 2\sqrt{10})} - \sqrt{(7 - 2\sqrt{10})}]^2$.

28. $[\sqrt{a + \sqrt{(a^2 - b^2)}} + \sqrt{a - \sqrt{(a^2 - b^2)}}]^2$.

29. $(\sqrt{2} + \sqrt{5} + \sqrt{7})(\sqrt{2} + \sqrt{5} - \sqrt{7})$.

30. $(\sqrt{31} + 2\sqrt{7} - 1)(\sqrt{31} - 2\sqrt{7} + 1)$.

31. $\sqrt{(5 + \sqrt{7})} \times \sqrt{(2 - \sqrt{2})} \times \sqrt{(5 - \sqrt{7})} \times \sqrt{(2 + \sqrt{2})}$.

32. $\sqrt[3]{2 - \sqrt{(2 + \sqrt{3})}} \times \sqrt[3]{2 + \sqrt{(2 + \sqrt{3})}} \times \sqrt[3]{(2 + \sqrt{3})}$.

33. $\sqrt[5]{\sqrt{(x + 16)} + \sqrt{(x - 16)}} \times \sqrt[5]{\sqrt{(x + 16)} - \sqrt{(x - 16)}}$.

Simplify each of the following expressions:

34. $\sqrt{(a^2 - b^2)} \times \sqrt{\dfrac{a + b}{a - b}}$. 35. $\sqrt{(6x^2 - 6)} \times \sqrt{\dfrac{3x - 3}{2x + 2}}$.

36. $\dfrac{x^3 - 8z^3}{\sqrt{(x^3 + 2x^2z + 4xz^2)}} \times \dfrac{x^2}{x - 2z}\sqrt{\dfrac{xz}{x^2 + 2xz + 4z^2}}$.

37. $\left(x + \dfrac{p}{2} + \sqrt{\dfrac{p^2}{4} - q}\right)\left(x + \dfrac{p}{2} - \sqrt{\dfrac{p^2}{4} - q}\right)$.

Division of Surds.

27. Division of Monomial Surds. — The converse of the principle of Art. 10 (iii.) evidently holds. That is,

$$\frac{\sqrt[q]{a}}{\sqrt[q]{b}}=\sqrt[q]{\frac{a}{b}}.$$

Ex. 1. $\dfrac{\sqrt{8}}{\sqrt{2}}=\sqrt{\dfrac{8}{2}}=\sqrt{4}=2.$

Ex. 2. $\dfrac{3\sqrt[3]{a^2}}{4\sqrt[2]{(3\,a)}}=\dfrac{3\sqrt[6]{a^4}}{4\sqrt[6]{(3^3a^3)}}=\dfrac{3}{4}\sqrt[6]{\dfrac{a^4}{3^3a^3}}=\dfrac{3}{4}\sqrt[6]{\dfrac{3^3\cdot a}{3^6}}=\frac{1}{4}\sqrt[6]{(27\,a)}.$

We thus have the following method:

Reduce surds of different orders to equivalent surds of the same order.

Multiply the quotient of the coefficients by the quotient of the surd factors.

Simplify the result.

EXERCISES IX.

Simplify each of the following expressions:

1. $\sqrt{60}\div\sqrt{5}.$ **2.** $\sqrt{15}\div\sqrt{\frac{3}{5}}.$

3. $\sqrt{\frac{21}{2}}\div\sqrt{\frac{7}{6}}.$ **4.** $\sqrt[3]{32}\div\sqrt[3]{4}.$

5. $\sqrt{(45\,x^3)}\div\sqrt{(5\,x)}.$ **6.** $\sqrt[3]{(16\,a^2)}\div\sqrt[3]{(64\,a^4)}.$

7. $\sqrt{x}\div\sqrt[3]{x}.$ **8.** $\sqrt{x^3}\div\sqrt[3]{x^2}.$ **9.** $\sqrt{x}\div\sqrt[4]{x}.$

10. $\sqrt{30}\div\sqrt[3]{45}.$ **11.** $3\sqrt{5}\div\sqrt[4]{15}.$ **12.** $\sqrt[4]{72}\div\sqrt[3]{12}.$

13. $6\sqrt{2}\div\sqrt[3]{9}.$ **14.** $2\sqrt[3]{6}\div\sqrt[6]{2}.$ **15.** $3\sqrt[6]{96}\div\sqrt[3]{18}.$

16. $\sqrt{(14\,ab)}\div\sqrt[3]{(28\,a^2b^2)}.$ **17.** $\sqrt[3]{(15\,x^2y)}\div\sqrt[4]{(25\,xy^2)}.$

18. $(\sqrt{6}-5\sqrt{14})\div\sqrt{2}.$ **19.** $(3\sqrt{10}-4\sqrt{15})\div\sqrt{5}.$

20. $(\sqrt{6}-3\sqrt[4]{4})\div\sqrt[4]{2}.$ **21.** $(\sqrt[3]{3}-3\sqrt[6]{6})\div\sqrt[6]{3}.$

22. $(3\sqrt{20}+2\sqrt{15}-4\sqrt{5})\div\sqrt{10}.$

23. $(6\sqrt[3]{4}-8\sqrt[3]{36}-15\sqrt[3]{48})\div\sqrt[3]{18}.$

24. $\sqrt{(b^2-a^2)}\div\sqrt{(a+b)}.$ **25.** $\sqrt[3]{(a^2b-ab^2)}\div\sqrt[3]{(b^2-a^2)}.$

26. $x\sqrt{(xy+y^2)}\div y\sqrt{(x^2+xy)}.$ **27.** $(x^2-y^2)\div\sqrt{(x^2y+xy^2)}.$

28. To **Rationalize** a surd expression is to free it from irrational numbers.

Thus, $\sqrt[3]{4}$ is rationalized by multiplying it by $\sqrt[3]{2}$, since $\sqrt[3]{4} \times \sqrt[3]{2} = \sqrt[3]{8} = 2$.

29. The quotient of one surd divided by another, expressed as a fraction, may be simplified by *rationalizing its denominator.*

Ex. 1. $\dfrac{\sqrt{5}}{\sqrt{3}} = \dfrac{\sqrt{5} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \dfrac{\sqrt{15}}{3} = \frac{1}{3}\sqrt{15}.$

We thus have the following method:

Multiply the numerator and denominator by a factor which will rationalize the denominator.

Ex. 2. $$\frac{2\sqrt{a}}{\sqrt[3]{(4\,a^2)}} = \frac{2\sqrt{a} \times \sqrt[3]{(2\,a)}}{\sqrt[3]{(4\,a^2)} \times \sqrt[3]{(2\,a)}} = \frac{2\sqrt[6]{a^3} \times \sqrt[6]{(4\,a^2)}}{\sqrt[3]{(8\,a^3)}}$$
$$= \frac{2\sqrt[6]{(4\,a^5)}}{2\,a} = \frac{\sqrt[6]{(4\,a^5)}}{a}.$$

30. The Divisor a Binomial Quadratic Surd. — We express the quotient as a fraction and rationalize the denominator.

Ex. 1.
$$\frac{3\sqrt{2} + 2\sqrt{3}}{5\sqrt{2} + 4\sqrt{3}} = \frac{(3\sqrt{2} + 2\sqrt{3})(5\sqrt{2} - 4\sqrt{3})}{(5\sqrt{2} + 4\sqrt{3})(5\sqrt{2} - 4\sqrt{3})}$$
$$= \frac{30 - 2\sqrt{6} - 24}{(5\sqrt{2})^2 - (4\sqrt{3})^2} = \frac{6 - 2\sqrt{6}}{50 - 48} = 3 - \sqrt{6}.$$

We thus have the following method:

Multiply the numerator and denominator by the conjugate of the denominator.

Ex. 2.
$$\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}} = \frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}} \times \frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}}$$
$$= \frac{1+x+2\sqrt{(1-x^2)}+1-x}{(1+x)-(1-x)} = \frac{1+\sqrt{(1-x^2)}}{x}.$$

31. When the denominator contains three quadratic surds, a similar method may be employed.

Ex. 3.

$$\frac{\sqrt{2}}{2\sqrt{3}-\sqrt{2}+\sqrt{5}}=\frac{\sqrt{2}(2\sqrt{3}-\sqrt{2}-\sqrt{5})}{[(2\sqrt{3}-\sqrt{2})+\sqrt{5}][(2\sqrt{3}-\sqrt{2})-\sqrt{5}]}$$
$$=\frac{2\sqrt{6}-2-\sqrt{10}}{12-4\sqrt{6}+2-5}=\frac{2\sqrt{6}-2-\sqrt{10}}{9-4\sqrt{6}}$$
$$=\frac{(2\sqrt{6}-2-\sqrt{10})(9+4\sqrt{6})}{(9-4\sqrt{6})(9+4\sqrt{6})}$$
$$=\tfrac{1}{15}(9\sqrt{10}+8\sqrt{15}-10\sqrt{6}-30).$$

EXERCISES X.

Change each of the following fractions into an equivalent fraction with a rational denominator:

1. $\frac{1}{\sqrt{2}}$. **2.** $\frac{12}{5\sqrt{3}}$. **3.** $\frac{8}{3\sqrt[3]{4}}$. **4.** $\frac{10}{7\sqrt[4]{25}}$.

5. $\frac{x}{\sqrt{x}}$. **6.** $\frac{ax}{\sqrt[3]{(a^2x)}}$. **7.** $\frac{3}{\sqrt[4]{(ab^2c^3)}}$. **8.** $\frac{a}{\sqrt[n]{(x^{n-2}y^3)}}$.

9. $\frac{1}{2-\sqrt{3}}$. **10.** $\frac{12}{5+\sqrt{21}}$. **11.** $\frac{5}{4+\sqrt{11}}$.

12. $\frac{3}{5-2\sqrt{6}}$. **13.** $\frac{1+\sqrt{2}}{2-\sqrt{2}}$. **14.** $\frac{\sqrt{3}+\sqrt{7}}{5\sqrt{3}-3\sqrt{7}}$.

15. $\frac{5\sqrt{2}-4\sqrt{3}}{5\sqrt{2}+4\sqrt{3}}$. **16.** $\frac{3\sqrt{5}-2\sqrt{2}}{2\sqrt{5}-3\sqrt{2}}$. **17.** $\frac{a\sqrt{b}+b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$.

18. $\frac{1}{\sqrt{(\sqrt{5}+2)}-\sqrt{(\sqrt{5}-2)}}$. **19.** $\frac{1+2\sqrt{(2a-1)}}{1-\sqrt{(2a-1)}}$.

20. $\frac{1}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$. **21.** $\frac{3+4\sqrt{3}}{\sqrt{6}+\sqrt{2}-\sqrt{5}}$.

22. $\frac{\sqrt{a}+\sqrt{b}}{a+b+\sqrt{(ab)}}$. **23.** $\frac{\sqrt{(3a-b)}+\sqrt{(a-3b)}}{\sqrt{(3a-b)}-\sqrt{(a-3b)}}$.

Surd Factors.

32. The expression

$$x^2+2ax+a^2$$

is evidently the square of $x+a$. The third term of this expression may be obtained as follows:

$$a^2=\left(\frac{2a}{2}\right)^2.$$

That is, *the third term is the square of half the coefficient of x.*

Consequently, if to any binomial of the form $x^2 + 2ax$, we add the square of half the coefficient of x, the resulting trinomial will be the square of a binomial.

This step is called **completing the square.**

Thus, if to $x^2 + 6x$, we add $(\frac{6}{2})^2, = 9$,

we have $$x^2 + 6x + 9, = (x+3)^2.$$

33. By applying the principle of the preceding article, we can transform an expression of the second degree into the difference of two squares, and hence factor it.

Ex. 1. Factor $x^2 + 6x + 7$.

We first complete $x^2 + 6x$ to the square of a binomial by adding $(\frac{6}{2})^2, = 9$. In order that the value of the expression may remain unchanged, we also subtract 9 from it. We then have

$$\begin{aligned} x^2 + 6x + 9 - 9 + 7, &= (x+3)^2 - 2 \\ &= (x+3)^2 - (\sqrt{2})^2 \\ &= (x + 3 + \sqrt{2})(x + 3 - \sqrt{2}). \end{aligned}$$

Ex. 2. Factor $x^2 + x - 1$.

We have $$\begin{aligned} x^2 + x - 1 &= x^2 + x + (\tfrac{1}{2})^2 - (\tfrac{1}{2})^2 - 1 \\ &= (x + \tfrac{1}{2})^2 - \tfrac{5}{4} \\ &= (x + \tfrac{1}{2})^2 - (\tfrac{1}{2}\sqrt{5})^2 \\ &= (x + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})(x + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}). \end{aligned}$$

Ex. 3. Factor $3x^2 + 4xy - 2y^2$.

Since the coefficient of x^2 is not 1, we first take out the factor 3. We then have

$$3x^2 + 4xy - 2y^2 = 3(x^2 + \tfrac{4}{3}xy - \tfrac{2}{3}y^2).$$

Completing $x^2 + \frac{4}{3}xy$ to the square of a binomial by adding $(\frac{2}{3}y)^2, = \frac{4}{9}y^2$, to the expression *within the parentheses*, and also subtracting $\frac{4}{9}y^2$ from it, we have

$$\begin{aligned} &3(x^2 + \tfrac{4}{3}xy + \tfrac{4}{9}y^2 - \tfrac{4}{9}y^2 - \tfrac{2}{3}y^2) \\ &\quad = 3[(x + \tfrac{2}{3}y)^2 - (\tfrac{1}{3}\sqrt{10}\,y)^2] \\ &\quad = 3(x + \tfrac{2}{3}y + \tfrac{1}{3}\sqrt{10}\,y)(x + \tfrac{2}{3}y - \tfrac{1}{3}\sqrt{10}\,y). \end{aligned}$$

We thus derive the following method:

If the coefficient of x^2 is 1, add to, and subtract from, the given expression the square of half the coefficient of x.

Write this result in the form $a^2 - b^2$ and factor.

If the coefficient of x^2 is not 1, factor out this coefficient, and treat the remaining factor as before.

EXERCISES XI.

Factor each of the following expressions:

1. $x^2 + 4x + 1$.	**2.** $x^2 - 2x - 11$.
3. $166 + 6x - x^2$.	**4.** $9x^2 + 12x - 1$.
5. $4x^2 - 4xy - 17y^2$.	**6.** $x^2 + \frac{2}{3}x - \frac{1}{9}$.
7. $2x^2 + 6x - 3$.	**8.** $3 + 2x - 11x^2$.
9. $x^2 - 2mx - 1$.	**10.** $m^2x^2 - 4mx + 4 - m^2n$.

Evolution of Surds.

34. The principle established in Ch. XIV., Art. 33, holds also for surds. For any positive number can be expressed as a power of a rational or irrational number, as in Art. 10.

We therefore have $\sqrt[pq]{a} = \sqrt[p]{\sqrt[q]{a}}$;

or, for present purposes, $\sqrt[p]{\sqrt[q]{a}} = \sqrt[pq]{a}$.

Ex. **1.** $\sqrt[4]{\sqrt[3]{5}} = \sqrt[12]{5}$.

It is important to notice that $\sqrt[p]{\sqrt[q]{a}} = \sqrt[q]{\sqrt[p]{a}}$.

Ex. **2.** $\sqrt[3]{\sqrt[5]{(8x^3)}} = \sqrt[5]{\sqrt[3]{(8x^3)}} = \sqrt[5]{(2x)}$.

Ex. **3.** $\sqrt[3]{[2x\sqrt{(ax)}]} = \sqrt[3]{(2x)} \times \sqrt[6]{(ax)} = \sqrt[6]{(4x^2)}\sqrt[6]{(ax)}$
$$= \sqrt[6]{(4ax^3)}.$$

We thus have the following method:

If possible, take the required root of the radicand; as in Ex. 2.

Otherwise, take the required root of the coefficient, and multiply the index of the surd by the index of the required root; as in Ex. 3.

Simplify the result.

EXERCISES XII.

Simplify each of the following expressions:

1. $\sqrt{\sqrt[3]{9}}$. 2. $\sqrt[4]{\sqrt[3]{16}}$. 3. $\sqrt[4]{\sqrt[3]{36}}$. 4. $\sqrt{(36\sqrt[3]{16})}$.

5. $\sqrt[4]{\sqrt[3]{a^8}}$. 6. $\sqrt[6]{\sqrt[3]{a^2}}$. 7. $\sqrt[3]{\sqrt[5]{(-x^3)}}$.

8. $\sqrt[3]{\sqrt[4]{(a^9x^{12})}}$. 9. $\sqrt[4]{(2a\sqrt[3]{a^2})}$. 10. $\sqrt[3]{(a\sqrt{a})}$.

11. $\sqrt[m]{\sqrt[n]{a^m}}$. 12. $\sqrt{\sqrt[3]{(\frac{25}{49}a^2b^6c^8)}}$. 13. $\sqrt[5]{(a^2\sqrt{a})}$.

14. $\sqrt{\dfrac{2}{\sqrt[3]{2}}}$. 15. $\sqrt[3]{\dfrac{a^2}{\sqrt{a}}}$. 16. $\sqrt[n-1]{\dfrac{a}{\sqrt[n]{a}}}$.

Properties of Quadratic Surds.

35. The symbol of equality cancelled, $\neq$, is read *is not equal to*; as $2 \neq 4$.

36. *A quadratic surd cannot be equal to the sum of a rational number and another quadratic surd*; or

$$\sqrt{a} \neq b + \sqrt{c},$$

wherein $\sqrt{a}$ and $\sqrt{c}$ are surds, and b is rational.

For, if $$\sqrt{a} = b + \sqrt{c},$$

then squaring, $$a = b^2 + 2b\sqrt{c} + c.$$

Transposing, $$2b\sqrt{c} = a - b^2 - c.$$

Dividing by $2b$, $$\sqrt{c} = \frac{a - b^2 - c}{2b}.$$

This equation asserts that $\sqrt{c}$, an irrational number, is equal to $\dfrac{a - b^2 - c}{2b}$, a rational number. This is a contradiction of terms, and therefore the hypothesis $\sqrt{a} = b + \sqrt{c}$ is untenable.

37. *If* $$a + \sqrt{b} = x + \sqrt{y}, \qquad (1)$$

wherein $\sqrt{b}$ and $\sqrt{y}$ are surds, and a and x are rational, then $a = x$ and $b = y$.

For, if $a \neq x$, let $a = x + m$.

Then (1) becomes $x + m + \sqrt{b} = x + \sqrt{y}$,

or $$m + \sqrt{b} = \sqrt{y}.$$

But, by Art. 36, this is impossible, unless $m = 0$.

When $m = 0$, $a = x$, and therefore $\sqrt{b} = \sqrt{y}$.

38. *If* $\sqrt{(a+\sqrt{b})} = \sqrt{x} + \sqrt{y}$, *then* $\sqrt{(a-\sqrt{b})} = \sqrt{x} - \sqrt{y}$.

From $$\sqrt{(a+\sqrt{b})} = \sqrt{x} + \sqrt{y},$$

we obtain $$a + \sqrt{b} = x + y + 2\sqrt{(xy)}.$$

Whence, by Art. 37, $$a = x + y, \tag{1}$$

and $$\sqrt{b} = 2\sqrt{(xy)}. \tag{2}$$

Subtracting (2) from (1),

$$a - \sqrt{b} = x + y - 2\sqrt{(xy)} = (\sqrt{x} - \sqrt{y})^2.$$

Therefore $$\sqrt{(a-\sqrt{b})} = \sqrt{x} - \sqrt{y}.$$

Square Roots of Simple Binomial Surds.

39. Ex. **1.** Find a square root of $3 + 2\sqrt{2}$.

Let $$\sqrt{(3+2\sqrt{2})} = \sqrt{x} + \sqrt{y}. \tag{1}$$

Then, by Art. 38, $$\sqrt{(3-2\sqrt{2})} = \sqrt{x} - \sqrt{y}. \tag{2}$$

Multiplying (1) by (2), $\sqrt{(9-8)} = x - y$,

or $$x - y = 1. \tag{3}$$

Squaring (1), $3 + 2\sqrt{2} = x + y + 2\sqrt{(xy)}$;

whence, by Art. 37, $$x + y = 3. \tag{4}$$

Solving (3) and (4), we have $x = 2$, $y = 1$.

Therefore $\sqrt{(3+2\sqrt{2})} = \sqrt{2} + \sqrt{1} = \sqrt{2} + 1$.

This example could have been solved by inspection. We change $3 + 2\sqrt{2}$ into the form

$$m + 2\sqrt{(mn)} + n = (\sqrt{m} + \sqrt{n})^2.$$

We then have

$$\sqrt{(3+2\sqrt{2})} = \sqrt{(2+2\sqrt{2}+1)} = \sqrt{(\sqrt{2}+1)^2} = \sqrt{2} + 1.$$

Ex. **2.** Solve, by inspection, $\sqrt{(21 - 3\sqrt{24})}$.

We have
$$\begin{aligned}\sqrt{(21-3\sqrt{24})} &= \sqrt{(21-2\sqrt{54})}\\ &= \sqrt{(18-2\sqrt{54}+3)}\\ &= \sqrt{(\sqrt{18}-\sqrt{3})^2}\\ &= \sqrt{18} - \sqrt{3} = 3\sqrt{2} - \sqrt{3}.\end{aligned}$$

In solving by inspection, first write the surd term of the given binomial surd in the form $2\sqrt{(mn)}$, *as* $3\sqrt{24} = 2\sqrt{54}$.

Then find by inspection two numbers whose sum is equal to the rational term of the given binomial surd, and whose product is equal to mn.

EXERCISES XIII.

Find a square root of each of the following expressions:

1. $7 + \sqrt{48}$.	**2.** $5 - \sqrt{24}$.	**3.** $2 + \sqrt{3}$.
4. $1\frac{1}{2} + \sqrt{2}$.	**5.** $3 - \sqrt{5}$.	**6.** $6 + \sqrt{11}$.
7. $8 - \sqrt{28}$.	**8.** $6 + 4\sqrt{2}$.	**9.** $7 + 2\sqrt{10}$.
10. $11 - 6\sqrt{2}$.	**11.** $11 + 4\sqrt{7}$.	**12.** $30 - 10\sqrt{5}$.
13. $\frac{5}{7} + \frac{1}{7}\sqrt{21}$.	**14.** $\frac{9}{11} - \frac{4}{11}\sqrt{2}$.	**15.** $\frac{21}{22} - \frac{2}{11}\sqrt{5}$.

16. $4a + 2\sqrt{(4a^2 - b^2)}$. **17.** $n - 2\sqrt{(n-1)}$.

18. $10n^2 + 1 - 6n\sqrt{(n^2+1)}$. **19.** $a - x - 2\sqrt{(a - x - 1)}$.

Approximate Values of Surd Numbers.

40. An approximate value of a surd number can be found to any degree of accuracy by the methods given in Ch. XIV.

Ex. **1.** Find an approximate value of $\sqrt{2}$ correct to three decimal places. The work proceeds as follows:

$$
\begin{array}{r|l}
2.00'00'00'00 & \underline{1.4142} \\
\underline{1} & 2 \\
1\,00 & \\
\underline{96} & \underline{24} \\
4\,00 & \\
\underline{2\,81} & \underline{281} \\
1\,19\,00 & \\
\underline{1\,12\,96} & \underline{2824} \\
6\,04\,00 & 2828
\end{array}
$$

The work is simplified by neglecting the decimal point, writing it only in the result. It is necessary to find the root to four decimal places in order to determine whether to take the figure found in the third place or the next greater figure, according to the well-known principle of Arithmetic.

We now have $\sqrt{2} = 1.4142 \cdots$.

This value lies between 1.4142, $= \frac{14142}{10000}$, and 1.4143, $= \frac{14143}{10000}$. It therefore differs from either of these fractions by less than they differ from each other.

But $$\tfrac{14143}{10000} - \tfrac{14142}{10000} = \tfrac{1}{10000}.$$

Consequently the error of taking either 1.4142 or 1.4143 as an approximate value of $\sqrt{2}$ is less than $\frac{1}{10000}$. By taking the root to more decimal places a still more accurate value can be found. It is therefore possible to find an approximate value such that the error will be less than any assigned number, however small.

Ex. 2. Find the value of $\sqrt[3]{(1-x)}$ to three terms.

The work proceeds as follows:

$$
\begin{array}{l|l}
1-x & \underline{1-\frac{1}{3}x-\frac{1}{9}x^2} \\
\underline{1} & 3\times 1^2 = 3 \\
-x & 3\times 1^2 + 3\times 1\times(-\frac{1}{3}x) + (-\frac{1}{3}x)^2 = 3 - x + \frac{1}{9}x^2 \\
\underline{-x+\frac{1}{3}x^2-\frac{1}{27}x^3} & \\
\hline
\quad -\frac{1}{3}x^2+\frac{1}{27}x^3 &
\end{array}
$$

An approximate value of a fractional surd is obtained most simply by rationalizing its denominator, then finding the required root of the numerator of the resulting fraction, and dividing this value by the denominator.

Ex. 3. Find an approximate value of $\dfrac{3}{\sqrt{2}}$, correct to three decimal places.

We have $$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} = \frac{3}{2}(1.4142) = 2.121.$$

EXERCISES XIV.

Find an approximate value of each of the following expressions, correct to four figures:

1. $\sqrt{8}$. 2. $\frac{1}{2}\sqrt{2.5}$. 3. $\sqrt{.2}$. 4. $\frac{3}{8}\sqrt{1.25}$.

5. $\sqrt{345.06}$. 6. $\sqrt{10862.321}$. 7. $\sqrt{54.0001}$.

8. $\dfrac{2}{\sqrt{5}}$. 9. $\dfrac{3}{\sqrt{8}}$. 10. $\dfrac{1}{2\sqrt[3]{4}}$. 11. $\dfrac{5}{\sqrt{75}}$.

12. $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$. **13.** $\dfrac{3+2\sqrt{7}}{5-4\sqrt{11}}$. **14.** $\dfrac{\sqrt{17}}{\sqrt{2.5}+\sqrt{6}}$.

Find an approximate value of each of the following expressions, to include four terms:

15. $\sqrt{(1-x)}$. **16.** $\sqrt{(a^2+b^2)}$. **17.** $\sqrt{(x^2-xy+y^2)}$.

18. $\sqrt[3]{(1+x^3)}$. **19.** $\sqrt[3]{(a^3-b^3)}$. **20.** $\sqrt[3]{(x^3+x^2y+xy^2+y^3)}$.

IRRATIONAL EQUATIONS.

41. An **Irrational Equation** is an equation whose members, either or both, are irrational in the unknown number or numbers; as

$$\sqrt{(x+1)}=3.$$

42. To solve an irrational equation, we must first derive from it a rational, integral equation. This step, which is usually effected by raising both members of the equation to the same positive integral power one or more times, is called *rationalizing the equation.*

Ex. **1.** Solve the equation $\sqrt{(36+x^2)}-x=2$.

Transferring x, $\sqrt{(36+x^2)}=2+x$.

Equating squares of both members,

$$36+x^2=4+4x+x^2.$$

Transferring and uniting terms,

$$-4x=-32.$$

Dividing by -4, $x=8$.

Check: $\sqrt{(36+64)}=2+8$, or $\sqrt{100}=10$.

Ex. **2.** Solve the equation $\sqrt{(45+x)}+\sqrt{x}=9$.

Transferring $\sqrt{x}$, $\sqrt{(45+x)}=9-\sqrt{x}$.

Equating squares, $45+x=81-18\sqrt{x}+x$.

Transferring and uniting terms,

$$18\sqrt{x}=36.$$

Dividing by 18 and equating squares,

$$x=4.$$

Check: $\sqrt{(45+4)}+\sqrt{4}=9$, or $7+2=9$.

The preceding examples illustrate the following method of solving irrational equations:

Transform the given equation so that one radical stands by itself in one member of the equation.

Equate equal powers of the two members when so transformed.

Repeat this process until a rational equation is obtained.

EXERCISES XV.

Solve each of the following equations:

1. $\sqrt{x}=5.$ **2.** $2\sqrt[3]{x}=3.$ **3.** $a\sqrt[n]{x}=b.$

4. $\sqrt{(x-1)}=5.$ **5.** $\sqrt{(7-x)}=2\sqrt{3}.$ **6.** $\sqrt[3]{(5x-7)}=2.$

7. $8-\sqrt{x}=4.$ **8.** $9=\sqrt{(3x)}+3.$ **9.** $a=\sqrt{x}+c.$

10. $\dfrac{\sqrt{x}+5}{\sqrt{x}-3}=5.$ **11.** $\dfrac{\sqrt{x}-8}{1-\sqrt{x}}=\dfrac{4}{3}.$ **12.** $\dfrac{\sqrt{(ax)}}{\sqrt{(ax)}-1}=\dfrac{a}{b}.$

13. $\sqrt{(7x+2)}=\dfrac{5x+6}{\sqrt{(7x+2)}}.$ **14.** $\sqrt{(x+5)}=\dfrac{x-1}{\sqrt{(x-3)}}.$

15. $9-\sqrt{(3x+1)}=5.$ **16.** $\sqrt{(7-3x)}=\sqrt{(9-4x)}.$

17. $2\sqrt{(x-7)}=\sqrt{(3x-17)}.$ **18.** $\sqrt[3]{(4x+9)}=\sqrt[3]{(7x-6)}.$

19. $\sqrt{x}+\sqrt{(5+x)}=5.$ **20.** $\sqrt{x}=11-2\sqrt{(7+x)}.$

21. $\sqrt{(36+x)}=2+\sqrt{x}.$ **22.** $\sqrt{x}-\sqrt{(x+9)}=-1.$

23. $a=\sqrt{(a+x)}-\sqrt{x}.$ **24.** $\sqrt[3]{(x^3+12x^2)}=x+4.$

25. $\sqrt{(6+x)}+\sqrt{(3+x)}=3.$

26. $3\sqrt{(x-3)}+\sqrt{(9x+1)}=14.$

27. $\sqrt{(2+\sqrt{x})}+\sqrt{(2-\sqrt{x})}=\sqrt{x}.$

28. $\sqrt{(14-x)}+\sqrt{(11-x)}=\dfrac{3}{\sqrt{(11-x)}}.$

29. $\dfrac{5+3\sqrt{(x-7)}}{1-6\sqrt{(x-7)}}=\dfrac{2\sqrt{(x-7)}-3}{7-4\sqrt{(x-7)}}.$

30. $\sqrt{(16x-15)}-\sqrt{(9x-11)}=\sqrt{x}.$

31. $\sqrt{(x+2a)}-\sqrt{(x+2b)}=2\sqrt{x}.$

32. $\sqrt{(x+4)}+\sqrt{(x-4)}=\sqrt{(4x-4)}.$

33. $\sqrt{(x+2)}+\sqrt{(x-6)}=2\sqrt{(x-3)}.$

34. $\sqrt{(x-5)}-\sqrt{(x+3)}=\sqrt{(x-2)}-\sqrt{(x+10)}.$

CHAPTER XVI.

IMAGINARY AND COMPLEX NUMBERS.

1. Attention was called in Ch. XIV., Art. 10, to the fact that $\sqrt{-16}$ cannot be expressed in terms of numbers with which we are, as yet, familiar. In general, since even powers of both positive and negative numbers are *positive*, even roots of negative numbers cannot be expressed in terms of either rational or irrational numbers.

It is therefore necessary either to exclude from our consideration such roots as $\sqrt{-1}$, and in general $\sqrt[2n]{-a}$, or again to enlarge our ideas of number.

2. We will now define, that is, fix the meaning of, the numbers $\sqrt{-1}$ and $\sqrt[2n]{-a}$, by assuming that they obey the law

$$(\sqrt[q]{a})^q = a.$$

This relation follows from the definition of a root, as was shown in Ch. XIV., Art. 5.

We therefore have $(\sqrt{-1})^2 = -1$, and $(\sqrt[2n]{-a})^{2n} = -a$.

Whatever meaning and use these new numbers have must be derived from these relations.

Imaginary Numbers.

3. The square root of a negative number is called an **Imaginary Number**; as $\sqrt{-3}$, $\sqrt{-8}$.

The study of these numbers is simplified by first considering the properties of $\sqrt{-1}$, which is taken as the **Imaginary Unit.***

* The designation, *imaginary*, is unfortunate, since, as will be shown in **Part II., Text-Book of Algebra**, such numbers are no more imaginary (in the ordinary meaning of the word) than common fractions or negative numbers. Dr. George Bruce Halsted, Professor of Mathematics in the University of Texas, has suggested **Neomon** for the *imaginary unit*, and **Neomonic** for *imaginary*.

This new unit is commonly designated by the letter i, and its opposite by $-i$.

We then have by definition

$$(\sqrt{-1})^2 = (\pm i)^2 = -1.$$

For the sake of distinction all numbers, rational and irrational, which have been used hitherto in this book are called **Real Numbers.**

4. The Fundamental Operations with the Imaginary Unit. — The imaginary unit $\sqrt{-1}$, or i, is used like a real term or factor in the fundamental operations.

Just as $\qquad 3 = 1 + 1 + 1$, and $-3 = -1 - 1 - 1$;

so $\quad 3\sqrt{-1} = \sqrt{-1} + \sqrt{-1} + \sqrt{-1}$, $\quad$ or $\quad 3i = i + i + i$;

$-3\sqrt{-1} = -\sqrt{-1} - \sqrt{-1} - \sqrt{-1}$, or $-3i = -i - i - i$.

Again,

$\sqrt{-1} \times 2 = 2\sqrt{-1}$, or $i\,2 = 2i$; $\dfrac{a\sqrt{-1}}{\sqrt{-1}} = a$, or $\dfrac{ai}{i} = a$.

5. We now have, in addition to the double series of real numbers, the double series of imaginary numbers:

$$\cdots -3i,\ -2i,\ -i,\ 0,\ i,\ 2i,\ 3i,\ \cdots.$$

6. Powers of i. — The following values of the positive integral powers of $\sqrt{-1}$, or i, follow directly from the definition of i and Art. 4:

$$
\begin{array}{ll}
\sqrt{-1} = \sqrt{-1}, & \text{or } i = i, \\
(\sqrt{-1})^2 = -1, & i^2 = -1, \\
(\sqrt{-1})^3 = (\sqrt{-1})^2(\sqrt{-1}) = -\sqrt{-1}, & i^3 = i^2 \cdot i = -i, \\
(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = +1, & i^4 = i^2 \cdot i^2 = +1, \\
(\sqrt{-1})^5 = (\sqrt{-1})^4(\sqrt{-1}) = +\sqrt{-1}, & i^5 = i^4 \cdot i = +i, \\
(\sqrt{-1})^6 = (\sqrt{-1})^4(\sqrt{-1})^2 = -1, & i^6 = i^4 \cdot i^2 = -1.
\end{array}
$$

The preceding results give the following properties of powers of i:

(i.) *All even powers of i are real.*

(ii.) *All odd powers of i are imaginary.*

The sign of any particular power of i is readily determined by expressing it as a power of i^2 if an *even* power, or of i^2 multiplied by i if an *odd* power.

Ex. 1. $i^{22} = (i^2)^{11} = (-1)^{11} = -1.$

Ex. 2. $i^{36} = (i^2)^{18} = (-1)^{18} = +1.$

Ex. 3. $i^{41} = i^{40} \times i = (i^2)^{20} \cdot i = (-1)^{20} \cdot i = +i.$

Ex. 4. $i^{39} = i^{38} \times i = (i^2)^{19} \cdot i = (-1)^{19} \cdot i = -i.$

7. Multiples of the Imaginary Unit. — Since

$$(\sqrt{-a})^2 = -a, \text{ and } (\sqrt{a} \times \sqrt{-1})^2 = (\sqrt{a})^2(\sqrt{-1})^2 = -a,$$

we have $$(\sqrt{-a})^2 = (\sqrt{a} \times \sqrt{-1})^2.$$

Whence $$\boldsymbol{\sqrt{-a} = \sqrt{a} \times \sqrt{-1}.}$$

Ex. 1. $\sqrt{-9} = \sqrt{9} \times \sqrt{-1} = 3\sqrt{-1} = 3i.$

Ex. 2. $\sqrt{-2} = \sqrt{2} \times \sqrt{-1} = \sqrt{2} \cdot i = i\sqrt{2}.$

In all reductions involving imaginary terms or factors it is advisable thus to express them as multiples of $\sqrt{-1}$ or i.

8. Addition of Imaginary Numbers. — Imaginary numbers are united by addition and subtraction just as real numbers are united.

Ex. 1. $\sqrt{-9} + \sqrt{-16} = 3\sqrt{-1} + 4\sqrt{-1} = 7\sqrt{-1} = 7i.$

Ex. 2. $10\sqrt{-5} - 4\sqrt{-5} = 6\sqrt{-5} = 6\sqrt{5} \times \sqrt{-1}$
$= 6i\sqrt{5}.$

Ex. 3. $i^{13} + i^{15} = i + (-i) = 0.$

9. Multiplication of Imaginary Numbers. — The principle of Art. 7 is of importance in the multiplication of imaginary numbers.

Ex. 1. $\sqrt{-9} \times \sqrt{16} = \sqrt{9} \times \sqrt{-1} \times \sqrt{16} = 12\sqrt{-1} = 12i.$

Ex. 2. $\sqrt{-2} \times \sqrt{-8} = \sqrt{2} \times \sqrt{-1} \times \sqrt{8} \times \sqrt{-1}$
$= \sqrt{16} \times (\sqrt{-1})^2 = -4.$

A point in Ex. 2 deserves special notice. Had we used the principle

$$\sqrt{a} \times \sqrt{b} = \sqrt{(ab)},$$

as in surds, we should have obtained

$$\sqrt{[(-2) \times (-8)]} = \sqrt{16} = 4, \text{ and not } -4.$$

But that principle was proved for *positive roots of positive numbers*, and therefore cannot be applied in this and similar examples.

Ex. 3.

$$\sqrt{-5} \times \sqrt{-10} \times \sqrt{-15} = \sqrt{5} \times \sqrt{10} \times \sqrt{15} \times (\sqrt{-1})^3$$
$$= -5\sqrt{30} \times \sqrt{-1} = -5\,i\sqrt{30}.$$

10. Division of Imaginary Numbers. — The following examples illustrate all possible cases.

Ex. 1. $\dfrac{\sqrt{-8}}{\sqrt{2}} = \dfrac{\sqrt{8} \times \sqrt{-1}}{\sqrt{2}} = \sqrt{\dfrac{8}{2}} \times \sqrt{-1} = 2\sqrt{-1} = 2\,i.$

Ex. 2. $\dfrac{1}{\sqrt{-1}} = \dfrac{\sqrt{-1}}{(\sqrt{-1})^2} = \dfrac{\sqrt{-1}}{-1} = -\sqrt{-1}$, or $\dfrac{1}{i} = -i.$

Ex. 3.

$$\frac{\sqrt{6}}{\sqrt{-3}} = \frac{\sqrt{6}}{\sqrt{3} \times \sqrt{-1}} = \frac{\sqrt{6} \times \sqrt{-1}}{\sqrt{3} \times (\sqrt{-1})^2} = -\sqrt{\frac{6}{3}} \times \sqrt{-1}$$
$$= -\sqrt{2} \times \sqrt{-1} = -i\sqrt{2}.$$

Ex. 4. $\dfrac{\sqrt{-9}}{\sqrt{-4}} = \dfrac{\sqrt{9} \times \sqrt{-1}}{\sqrt{4} \times \sqrt{-1}} = \dfrac{\sqrt{9}}{\sqrt{4}} = \dfrac{3}{2}.$

EXERCISES I.

Reduce each of the following expressions to the form $a\sqrt{-1}$, or ai:

1. $\sqrt{-9}$. 2. $6\sqrt{-25}$. 3. $\sqrt{-a^2}$. 4. $a\sqrt{-x^{2n}}$.
5. $\sqrt{-12}$. 6. $\sqrt{-10}$. 7. $\sqrt{-x^3}$. 8. $\sqrt{(-5\,a^3)}$.
9. $\sqrt{(-8\,a^3b^3)}$. 10. $\sqrt{(3-27)}$. 11. $\sqrt[6]{-64}$. 12. $\sqrt[6]{-a^{12}}$.

Simplify each of the following expressions:

13. i^8. **14.** i^{29}. **15.** i^{54}. **16.** $i^4 + i^{34}$.

17. $\dfrac{1}{i^6}$. **18.** $\dfrac{1}{i^{17}}$. **19.** $\dfrac{1}{i^{52}}$. **20.** $\dfrac{1}{i^{39}+i^{55}}$.

Add:

21. $\sqrt{-9} + \sqrt{-25}$. **22.** $\sqrt{-16} - \sqrt{-121}$.

23. $\sqrt{-a^2} - \sqrt{-b^2}$. **24.** $7\sqrt{-81} + 5\sqrt{-144}$.

25. $5\sqrt{-8} - 3\sqrt{-32}$. **26.** $8\sqrt{-75} + \sqrt{-147}$.

27. $2\sqrt{-25} - 3\sqrt{-49} + 4\sqrt{-100}$.

28. $2\sqrt{-a^2} + 5\sqrt{(-9a^2)} - 3\sqrt{(-16a^2)}$.

29. $2\sqrt{(-a^4b)} - 4\sqrt{(-a^2b^3)} + 2\sqrt{-b^5}$.

Perform the following indicated operations:

30. $\sqrt{-x^4}$. **31.** $(\sqrt{-x})^4$. **32.** $(\sqrt{-a})^8$. **33.** $\sqrt{-a^8}$.

34. $\sqrt{3} \times \sqrt{-6}$. **35.** $\sqrt{-2} \times \sqrt{-8}$. **36.** $\sqrt{-12} \times \sqrt{3}$.

37. $\sqrt{-2} \times \sqrt{-50}$. **38.** $\sqrt{-a} \times \sqrt{(-9a^3)}$.

39. $\sqrt{-x^3} \times \sqrt[6]{-x^6}$. **40.** $\sqrt{-6} \times \sqrt{12}$.

41. $\sqrt{-8} \times \sqrt{-20}$. **42.** $\sqrt{-x^2} \times \sqrt{-y^4}$.

43. $\sqrt{-2} \times \sqrt{-6} \times \sqrt{-24}$. **44.** $\sqrt{-5} \times \sqrt{8} \times \sqrt{-20}$.

45. $\sqrt{(1-x)} \times \sqrt{(x-1)}$. **46.** $\sqrt{(b^2-a^2)} \times \sqrt{(a-b)}$.

47. $(\sqrt{-5} + \sqrt{-3})^2$. **48.** $(2\sqrt{-3} + 3\sqrt{-2})^2$.

49. $\sqrt{-3} \div \sqrt{-3}$. **50.** $\sqrt{-3} \div \sqrt{3}$. **51.** $\sqrt{3} \div \sqrt{-3}$.

52. $\sqrt{-8} \div \sqrt{-2}$. **53.** $\sqrt{-75} \div \sqrt{5}$. **54.** $\sqrt{12} \div \sqrt{-3}$.

Complex Numbers.

11. A **Complex Number** is the algebraic sum of a real and an imaginary number; as, $3 \pm 2i$.

The general form of a complex number is evidently $a + bi$, wherein a and b are real numbers.

When $b = 0$, we have any real number.

When $a = 0$, we have any imaginary number.

12. *Two complex numbers are said to be equal when the real term of one is equal to the real term of the other, and the imaginary term of one is equal to the imaginary term of the other;* as, $2+3i=2+3i$.

That is, if $$a+bi=c+di,$$

then $$a=c, \text{ and } bi=di, \text{ or } b=d.$$

Observe that the preceding statement is a definition of the meaning of the sign of equality between two complex numbers.

13. From the preceding article it follows that, if

$$a+bi=0=0+0i, \text{ then } a=0,\ b=0.$$

14. Addition and Subtraction of Complex Numbers. — We define algebraic addition of two or more complex numbers as follows:

Add the real terms by themselves and the imaginary terms by themselves.

Ex. 1. $$\begin{aligned}(2+3\sqrt{-1})+(6\sqrt{-1}-5)&=(2-5)+(3+6)\sqrt{-1}\\ &=-3+9\sqrt{-1}=-3+9i.\end{aligned}$$

15. Multiplication of Complex Numbers. — We define multiplication of complex numbers by assuming that the distributive law holds.

Ex. 1. $$\begin{array}{l} 2+\ 3\sqrt{-1} \\ \underline{4-\ 2\sqrt{-1}} \\ 8+12\sqrt{-1} \\ \underline{\quad -\ 4\sqrt{-1}-6(\sqrt{-1})^2} \\ 8+\ 8\sqrt{-1}+6 \qquad\qquad =14+8\sqrt{-1}=14+8i. \end{array}$$

16. Conjugate Complex Numbers. — Two complex numbers which differ only in the sign of their imaginary terms are called **Conjugate Complex Numbers**; as,

$$2+3\sqrt{-1} \text{ and } 2-3\sqrt{-1},\quad -4+6i,\quad -4-6i.$$

17. *The sum of two conjugate complex numbers is real.*

Ex. 1. $(-2+3\sqrt{-1})+(-2-3\sqrt{-1})=-4.$

The product of two conjugate complex numbers is real and positive.

Ex. 2. $(4-5\sqrt{-1})(4+5\sqrt{-1})=4^2-(5\sqrt{-1})^2$

$$=16+25=41.$$

18. Division of Complex Numbers. — We express the quotient as a fraction, and simplify the result.

Ex. 1. $$\frac{1+\sqrt{-2}}{2\sqrt{-3}}=\frac{(1+\sqrt{-2})(\sqrt{-3})}{2(\sqrt{-3})^2}=\frac{\sqrt{-3}-\sqrt{6}}{-6}$$

$$=\tfrac{1}{6}\sqrt{6}-\tfrac{1}{6}\sqrt{-3}=\tfrac{1}{6}\sqrt{6}-\tfrac{1}{6}i\sqrt{3}.$$

Ex. 2. $$\frac{1}{2+\sqrt{-5}}=\frac{2-\sqrt{-5}}{(2+\sqrt{-5})(2-\sqrt{-5})}=\frac{2-\sqrt{-5}}{2^2-(\sqrt{-5})^2}$$

$$=\frac{2-\sqrt{-5}}{9}=\tfrac{2}{9}-\tfrac{1}{9}i\sqrt{5}.$$

19. Any Even Root of a Negative Number. — We have

$$(1+\sqrt{-1})^4=[(1+\sqrt{-1})^2]^2$$

$$=(1+2\sqrt{-1}-1)^2=(2\sqrt{-1})^2=-4.$$

Therefore, $\sqrt[4]{-4}=1+\sqrt{-1}.$

That is, the *fourth* root of -4 is a complex number.

It will be proved in Text-book of Algebra, Part II, that *any even* root of a negative number is a complex number.

Complex Factors.

20. A quadratic expression which is the product of two complex factors can be resolved into factors by the method used to resolve a quadratic expression into irrational factors.

Ex. Factor $x^2-2x+3.$

Completing x^2-2x to the square of a binomial in x, we have

$$x^2-2x+3=x^2-2x+1-1+3$$

$$=(x-1)^2-(\sqrt{-2})^2$$

$$=(x-1+\sqrt{-2})(x-1-\sqrt{-2}).$$

EXERCISES II.

Simplify each of the following expressions:

1. $(2+4i)+(2i-3)$.
2. $(7-5i)-(3-4i)$.
3. $(1+\sqrt{-9})+(4-\sqrt{-4})$.
4. $(6-\sqrt{-16})-(5-\sqrt{-36})$.
5. $(1+\sqrt{-1})(1-\sqrt{-1})$.
6. $(2+i\sqrt{3})(2-i\sqrt{3})$.
7. $(2+3\sqrt{-1})(3-4\sqrt{-1})$.
8. $(7+\sqrt{-5})(7-\sqrt{-5})$.
9. $(3+5i)(\sqrt{12}-3i)$.
10. $(\sqrt{8}-\sqrt{-12})(\sqrt{2}-\sqrt{-3})$.
11. $(\frac{1}{4}-\frac{1}{4}i\sqrt{3})(3+3i\sqrt{3})$.
12. $(5-2i\sqrt{6})(5+2i\sqrt{6})$.
13. $[x+i\sqrt{(a-x^2)}][x-i\sqrt{(a-x^2)}]$.

Perform the following indicated divisions:

14. $\dfrac{3}{1+\sqrt{-2}}$.
15. $\dfrac{7}{2-\sqrt{-3}}$.
16. $\dfrac{3+2\sqrt{-1}}{2-3\sqrt{-1}}$.
17. $\dfrac{1+i}{1-i}$.
18. $\dfrac{3+2i}{3-2i}$.
19. $\dfrac{5+i\sqrt{3}}{5-i\sqrt{3}}$.
20. $\dfrac{a+bi}{a-bi}$.

Factor each of the following expressions:

21. $x^2-6x+25$.
22. $x^2+4x+68$.
23. $x^2-14x+61$.
24. $5x^2-6x+2$.
25. $4x^2+4xy+3y^2$.
26. $16x^2-8xy+5y^2$.

Make the indicated substitution in each of the following expressions, and simplify the results:

27. In $x^2-6x+14$, let $x=3+\sqrt{-5}$.
28. In $3x^2-5x+7$, let $x=2-3\sqrt{-2}$.
29. In $x^2+2xy+y^2$, let $x=4+5i$, $y=4-5i$.

CHAPTER XVII.

DOCTRINE OF EXPONENTS.

1. We have already abbreviated such products as

$$aa,\ aaa,\ aaaa,\ \cdots,\ aaa \cdots n \text{ factors},$$

by a^2, a^3, a^4, $\cdots$, a^n, respectively, and called them the *second, third, fourth,* $\cdots$, *nth,* powers of a. This definition of the symbol a^n requires the exponent n to be a *positive integer.*

Thus 2^5 means the product of 5 factors, each equal to 2. But 2^0 has, as yet, no meaning, since 2 cannot be taken 0 times as a factor. For a similar reason 2^{-5} and $2^{\frac{1}{3}}$ are, as yet, meaningless.

But, having introduced into Algebra the symbol a^n, it is natural to inquire what it may mean when n is 0, *negative,* or a *fraction.*

We shall find that, by enlarging our conception of *powers,* quite clear and definite meanings can be given to such expressions as 2^0, 3^{-2}, and $4^{\frac{1}{2}}$.

Positive Integral Powers.

2. The principles

$$a^m \times a^n = a^{m+n},\quad a^m \div a^n = a^{m-n},$$

wherein m and n are positive integers, were illustrated by particular examples in Ch. III., Arts. 24 and 37.

In general,

$$\begin{aligned} a^m \times a^n &= (aaa \cdots \text{to } m \text{ factors})\,(aaa \text{ to } n \text{ factors}) \\ &= aaa \cdots \text{to } m+n \text{ factors} = a^{m+n}. \\ a^m \div a^n &= (aaa \cdots \text{to } m \text{ factors}) \div (aaa \cdots \text{to } n \text{ factors}) \\ &= [aaa \cdots \text{to } (m-n) \text{ factors}] \times (aaa \cdots \text{to } n \text{ factors}) \\ &\qquad \div (aaa \cdots \text{to } n \text{ factors}) \\ &= aaa \cdots \text{to } (m-n) \text{ factors},\quad = a^{m-n}. \end{aligned}$$

3. The other principles upon which operations with positive integral powers depend have been proved in Ch. XIII.

(i.) $$a^m a^n = a^{m+n}.$$

(ii.) $$\frac{a^m}{a^n} = a^{m-n}, \text{ when } m > n; \quad \frac{a^m}{a^n} = 1, \text{ when } m = n;$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \text{ when } m < n.$$

(iii.) $(a^m)^n = a^{mn}$. (iv.) $(ab)^m = a^m b^m$.

(v.) $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

Zeroth Powers.

4. The meaning of a symbol may be defined by assuming that it stands for the result of a definite operation, as was done in letting

$$a^n = a \cdot a \cdot a \cdot \cdots n \text{ factors};$$

or by enlarging the meaning of some operation or law which was previously restricted in its application.

In the latter way, negative numbers were introduced by extending the meaning of subtraction.

5. We now enlarge the meaning of powers by assuming that the principle

$$\frac{a^m}{a^n} = a^{m-n}$$

holds also when $m = n$.

We then have $$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

But since $$\frac{a^m}{a^m} = 1,$$

it follows that $$a^0 = 1.$$

That is, *the zeroth power of any base, except* 0, *is equal to* 1.

E.g., $1^0 = 1$, $5^0 = 1$, $99^0 = 1$, $(a + b)^0 = 1$, etc.

6. Thus, by the assumption that the stated law holds when $m = n$, a definite value of the zeroth power of a number is obtained. Nevertheless, it will doubtless seem strange to the student that all numbers to the zeroth power have one and the same value, namely 1. But it should be distinctly noted that a^0 is by definition a symbol for $\frac{a^m}{a^m}$; *i.e.*, for the quotient of two like powers of the same base. Thus,

$$2^0 = \frac{2^3}{2^3} = \frac{2^5}{2^5} = \frac{2^m}{2^m} = 1.$$

Negative Integral Powers.

7. We now still further enlarge the meaning of powers by assuming that the principle

$$\frac{a^m}{a^n} = a^{m-n}$$

holds not only when $m > n$ and $m = n$, but also when $m < n$. We then have, for example,

$$\frac{a^2}{a^5} = a^{2-5} = a^{-3}.$$

But, cancelling as in fractions,

$$\frac{a^2}{a^5} = \frac{1}{a^3}.$$

Therefore, $$a^{-3} = \frac{1}{a^3}.$$

In general, since $m < n$, we may assume $n = m + k$.

Then $$\frac{a^m}{a^n} = \frac{a^m}{a^{m+k}} = a^{m-(m+k)} = a^{-k}.$$

But $$\frac{a^m}{a^{m+k}} = \frac{1}{a^{m+k-m}} = \frac{1}{a^k}.$$

Therefore, $$a^{-k} = \frac{1}{a^k}.$$

That is, *a negative power of a number is equal to the reciprocal of a positive power of the same number, the exponents being numerically equal.*

E.g., $$\left(\frac{a}{b}\right)^{-2}=\frac{1}{\left(\frac{a}{b}\right)^2}=\frac{1}{\frac{a^2}{b^2}}=\frac{b^2}{a^2}=\left(\frac{b}{a}\right)^2.$$

8. We also have $$\frac{1}{a^{-k}}=\frac{1}{\frac{1}{a^k}}=a^k.$$

This relation and the relation which defined a negative integral power may be stated thus:

Any power of a number may be transferred from the denominator to the numerator, or from the numerator to the denominator, of a fraction, if the sign of its exponent be reversed.

E.g., $$\frac{a^2}{a^{-3}}=a^2\cdot a^3=a^5;\ \frac{(-a)^{-4}}{a}=\frac{1}{a(-a)^4}=\frac{1}{a^5}.$$

This reciprocal relation between positive and negative powers is useful in reductions which involve negative powers.

EXERCISES I.

Find the value of each of the following expressions:

1. 2^{-3}. **2.** 3^{-2}. **3.** $(\frac{2}{3})^{-1}$. **4.** $(3\frac{3}{4})^{-3}$.

5. $(\frac{1}{3})^{-3}$. **6.** $\frac{1}{.25^{-4}}$. **7.** $\frac{1}{.2^{-6}}$. **8.** $(2^0)^{-6}$.

Change each of the following expressions into an equivalent expression in which all the exponents are positive:

9. x^3y^{-4}. **10.** $2\,c^{-4}d$. **11.** $3^{-1}a^2n^{-3}$. **12.** $5\,x^{-2}y^{-3}$.

13. $\frac{2\,n^{-3}}{a^{-1}b^2}$. **14.** $\frac{4\,b^2}{a^{-5}c}$. **15.** $\frac{5\,ad^{-2}}{7^{-1}b^{-3}c}$. **16.** $\frac{3\,a^{-2}n^{-2}}{8\,b^{-4}}$.

In each of the following expressions transfer the factors from the denominator to the numerator:

17. $\frac{a}{b^2}$. **18.** $\frac{2\,x^2}{5\,y^{-3}}$. **19.** $\frac{3\,x^{-3}}{2^{-2}y}$. **20.** $\frac{5\,xy}{ab}$.

21. $\frac{3}{(a+b)}$. **22.** $\frac{4\,(x+y)^3}{(x-y)^2}$. **23.** $\frac{2\,a\,(x^2+1)}{3\,a^{-1}(x^2-1)^3}$.

24–30. Find the values of the expressions in Exx. 17–23, when $a=3$, $b=4$, $x=-2$, $y=5$.

Fractional (Positive or Negative) Powers.

9. We will define, *i.e.*, fix the meaning of, the power $a^{\frac{1}{q}}$, in which q is a positive integer, by assuming that it must obey the first law of exponents, namely,

$$a^m \cdot a^n = a^{m+n}.$$

In other words, whatever meaning $a^{\frac{1}{q}}$ may have must be derived by an application of this law.

By this law, $\quad a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$

But, since $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2$, by definition of positive integral power of *any* base, we have

$$(a^{\frac{1}{2}})^2 = a.$$

That is, $a^{\frac{1}{2}}$ is a number whose *square is a*, or $a^{\frac{1}{2}} = \sqrt{a}$.

In general,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots q \text{ factors} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\cdots q \text{ terms}} = a^{q \cdot \frac{1}{q}} = a.$$

But, since $a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots q$ factors $= (a^{\frac{1}{q}})^q$, by definition of positive integral power, we have $(a^{\frac{1}{q}})^q = a$.

That is, $a^{\frac{1}{q}}$ is a number whose *qth power is a*,

or $$\boldsymbol{a^{\frac{1}{q}} = \sqrt[q]{a}}.$$

We are thus led, by the definition of the fractional power, $a^{\frac{1}{q}}$, to the operation that is inverse to that of raising a number to a positive integral power, *i.e.*, to the operation of finding a root.

Thus, $9^{\frac{1}{2}}$ and $\sqrt{9}$, $(-243)^{\frac{1}{5}}$ and $\sqrt[5]{-243}$, $a^{\frac{1}{q}}$ and $\sqrt[q]{a}$, are only different ways of representing the same numbers.

Notice that the index of the root is the *denominator* of the exponent of the fractional power, and the radicand is the *base*.

10. From the definition of a fractional power we have

$$(9^{\frac{1}{2}})^2 = (\sqrt{9})^2 = 9,\ [(-25)^{\frac{1}{3}}]^3 = (\sqrt[3]{-25})^3 = -25.$$

In general, $$(a^{\frac{1}{q}})^q = (\sqrt[q]{a})^q = a.$$

Also, $$(a^q)^{\frac{1}{q}} = \sqrt[q]{a^q} = a,$$
if only positive roots be considered.

Therefore, $$(a^{\frac{1}{q}})^q = (a^q)^{\frac{1}{q}},$$
for the positive root.

11. *Meaning of $a^{\frac{p}{q}}$, wherein $\frac{p}{q}$ is a positive or a negative fraction.* We may always assume q to be positive and p to have the sign of the fraction.

Whatever meaning $a^{\frac{p}{q}}$ may have must be derived by an application of the law
$$a^m \cdot a^n = a^{m+n}.$$

By this law, $$5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} = 5^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = 5^2.$$

But, since $5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} = (5^{\frac{2}{3}})^3$, we have $(5^{\frac{2}{3}})^3 = 5^2$.

That is, $5^{\frac{2}{3}}$ is a number whose *cube is* 5^2; or $5^{\frac{2}{3}} = \sqrt[3]{5^2}$.
In general,

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots q \text{ terms}} = a^{q \cdot \frac{p}{q}},\ = a^p.$$

But, since $a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots q$ factors $= (a^{\frac{p}{q}})^q$, we have $(a^{\frac{p}{q}})^q = a^p$.

That is, $a^{\frac{p}{q}}$ is a number whose *qth power is a^p*;

or $$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Notice that a fractional power is a root of an integral power. The denominator of the fractional exponent is the index of the root, and the numerator is the exponent of the power.

E.g., $23^{\frac{4}{5}} = \sqrt[5]{23^4}$; $(-19)^{\frac{2}{3}} = \sqrt[3]{(-19)^2}$; $2^{-\frac{2}{3}} = \sqrt[3]{2^{-2}} = \sqrt[3]{\frac{1}{4}}$.

12. Since fractional powers simply afford another way of indicating roots, all the principles relating to roots which were proved in Chapters XIV. and XV. hold for such powers.

EXERCISES II.

Write each of the following expressions as an equivalent expression with radical signs:

1. $a^{\frac{1}{2}}$. **2.** $b^{-\frac{1}{4}}$. **3.** $x^{\frac{2}{3}}$. **4.** $3\,y^{\frac{1}{2}}$.

5. $4\,x^{-\frac{3}{2}}y^{\frac{1}{2}}$. **6.** $2\,ab^{-\frac{5}{6}}c$. **7.** $2^{-1}x^{\frac{3}{4}}y^{\frac{1}{5}}$. **8.** $2\,a^{\frac{m}{n}}b^{-\frac{p}{q}}$.

9. $\left(\dfrac{a}{b}\right)^{\frac{4}{5}}$. **10.** $\left(\dfrac{2\,x}{3\,y}\right)^{-\frac{5}{6}}$. **11.** $\dfrac{4\,m^{\frac{4}{3}}}{3\,n^{\frac{5}{6}}}$. **12.** $\dfrac{ab^{-\frac{m}{n}}}{xy^{-\frac{p}{q}}}$.

Find the value of each of the following expressions:

13. $4^{\frac{1}{2}}$. **14.** $169^{\frac{1}{2}}$. **15.** $16^{-\frac{1}{2}}$. **16.** $144^{-\frac{1}{2}}$.

17. $27^{\frac{1}{3}}$. **18.** $27^{-\frac{1}{3}}$. **19.** $16^{\frac{1}{4}}$. **20.** $81^{-\frac{1}{4}}$.

21. $49^{\frac{3}{2}}$. **22.** $512^{\frac{2}{3}}$. **23.** $216^{-\frac{5}{3}}$. **24.** $32^{-\frac{3}{5}}$.

Write each of the following expressions as an equivalent expression with fractional exponents:

25. $\sqrt{a}$. **26.** $\sqrt{a^3}$. **27.** $\sqrt{(a^{-3}b^7)}$. **28.** $\sqrt{(2\,xy^{-5})}$.

29. $\sqrt[3]{a^2}$. **30.** $\sqrt[3]{(2\,x^{-1}y^2)}$. **31.** $\sqrt[4]{(5\,x^{-2}y^5)}$. **32.** $\sqrt[5]{(3\,a^{-7}b^6)}$.

13. Having thus determined definite meanings for zeroth, negative, and fractional powers, it remains to prove that they obey all the principles of positive integral powers.

Products of Powers.

(I.) $$a^m a^n = a^{m+n},$$

for all rational values of m and n.

Ex. **1.** $$x^5x^{-7} = x^{5+(-7)} = x^{5-7} = x^{-2} = \frac{1}{x^2}.$$

Ex. **2.** $$a^{\frac{1}{2}}b^{-\frac{3}{4}} \times a^{-3}b^4 = a^{\frac{1}{2}-3}b^{-\frac{3}{4}+4} = a^{-\frac{5}{2}}b^{\frac{13}{4}} = \frac{b^{\frac{13}{4}}}{a^{\frac{5}{2}}}.$$

Assume m to be positive and n negative, and the absolute value of m less than the absolute value of n.

Let $n = -n_1$, so that n_1 is positive. Then

$$a^m a^n = a^m a^{-n_1} = \frac{a^m}{a^{n_1}} = \frac{1}{a^{n_1-m}} = \frac{1}{a^{-[m+(-n_1)]}} = a^{m+(-n_1)} = a^{m+n}.$$

In a similar way the principle can be proved for other cases in which the exponents are 0 or negative.

That the principle holds when the exponents, either or both, are fractions, follows from the definition of a fractional power.

EXERCISES III.

Simplify each of the following expressions:

1. x^3x^0. **2.** $x^{-3}x^3$. **3.** $a^{-5}a^6$. **4.** $m^{-3}m^{-5}$.

5. $a^3a^{\frac{1}{2}}$. **6.** $a^{\frac{2}{3}}a^{\frac{3}{4}}$. **7.** $b^{-\frac{5}{6}}b^{\frac{2}{3}}$. **8.** $c^{-\frac{1}{3}}c^{-\frac{3}{8}}$.

9. $5\,a^{-3} \times 3\,a^5$. **10.** $-\frac{5}{7}\,b^{-2} \times 1\frac{2}{5}\,b^{-3}$. **11.** $a^3b^{-2} \times a^{\frac{1}{3}}b^{\frac{2}{5}}$.

12. $\dfrac{12\,a^{-3}}{n^{-2}} \times \dfrac{a^2}{9\,n^3}$. **13.** $\dfrac{7\,c^{-3}}{3\,a^3} \div \dfrac{35\,a^{-4}}{6\,c^2}$. **14.** $\dfrac{a^{-n}b^{-n}}{\frac{1}{2}\,c} \div \dfrac{c^{-1}}{a^{-2n}b^{-2n}}$.

15. $(a^{\frac{1}{3}} + x^{-2})(a^{\frac{1}{3}} - x^{-2})$. **16.** $(a^{\frac{1}{2}} + a^{-\frac{1}{2}})(a^{\frac{1}{2}} - a^{-\frac{1}{2}})$.

17. $(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}})$. **18.** $(x^2y^{-\frac{2}{3}} + xy^{-\frac{1}{3}} + 1)(xy^{-\frac{1}{3}} - 1)$.

19. $(a^{-7} + a^{-5} - a^{-3})(a^7 + a^5 + a^3)$.

20. $(x^3 - x^{-3} - 2\,x^{-6} + 5)(10\,x^{-7} + x^{-1} - 5\,x^{-4})$.

21. $(x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}})(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$.

22. $(a^{\frac{2}{3}} + a^{-\frac{2}{3}} - a^{\frac{1}{3}} - a^{-\frac{1}{3}})(a^{\frac{1}{3}} + a^{-\frac{1}{3}} + 1)$.

23. $(x^{\frac{2}{3}} + 2\,x^5 + 3\,x^{\frac{1}{3}} + 2\,x^{\frac{1}{6}} + 1)(x^{\frac{1}{3}} - 2\,x^{\frac{1}{6}} + 1)$.

Quotients of Powers.

(II.) $$\frac{a^m}{a^n} = a^{m-n},$$

for all rational values of m and n.

Ex. 1. $$\frac{x^2}{x^{-3}} = x^{2-(-3)} = x^{2+3} = x^5.$$

Ex 2. $$\frac{a^{-\frac{1}{2}}b^{\frac{2}{3}}}{a^{\frac{1}{4}}b^{-\frac{3}{2}}} = a^{-\frac{1}{2}-\frac{1}{4}}b^{\frac{2}{3}+\frac{3}{2}} = a^{-\frac{3}{4}}b^{\frac{13}{6}} = \frac{b^{\frac{13}{6}}}{a^{\frac{3}{4}}}.$$

We have $$\frac{a^m}{a^n} = a^m a^{-n} = a^{m+(-n)} = a^{m-n}.$$

EXERCISES IV.

Simplify each of the following expressions:

1. $\dfrac{a}{a^{-1}}$. **2.** $\dfrac{x^0}{x^{-2}}$. **3.** $\dfrac{5^{-2}}{5^{-3}}$. **4.** $\dfrac{a^2}{a^{\frac{1}{2}}}$. **5.** $\dfrac{x^{-2}}{x^{-5}}$.

6. $\dfrac{a^{-\frac{3}{4}}}{a^{-\frac{1}{4}}}$. **7.** $\dfrac{a^{\frac{2}{5}}}{a^{-2}}$. **8.** $\dfrac{x^n}{x^{-n}}$. **9.** $\dfrac{x^{m-n}}{x^{-n}}$. **10.** $\dfrac{x^{-1}}{x^{n-1}}$.

11. $(1\frac{1}{2}\,b^{-3}) \div (3\,b^2)$. **12.** $1 \div (\frac{1}{2}\,ab^{-1})$.

13. $(3\frac{1}{2}\,a^n b^{-4}) \div (\frac{7}{8}\,a^n b^{-3})$. **14.** $(a^{\frac{1}{2}} - b^{\frac{1}{2}}) \div (a^{\frac{1}{4}} + b^{\frac{1}{4}})$.

15. $(x^{-1} + y^{-1}) \div (x^{-\frac{1}{3}} + y^{-\frac{1}{3}})$.

16. $(3\,a^{-10} + a^6 - 4\,a^{-6}) \div (2\,a^{-2} + a^2 + 3\,a^{-6})$.

17. $(2\,x^{-3} - 3\,x^{-2} - 2\,x^{-1} + 2 - x) \div (x^{-1} + 1)$.

18. $(x^{-1} - 3\,x^{-\frac{1}{2}} + 3 - 3\,x^{\frac{1}{2}} + 2\,x) \div (x^{-\frac{3}{2}} - 2\,x^{-1} + x^{-\frac{1}{2}} - 2)$.

19. $(2\,a^7 - 3\,a^3 - 23\,a^{-1} + 15\,a^{-5} + 9\,a^{-9}) \div (a^4 + 2 - 3\,a^{-4})$.

20. $(6\,x^{\frac{1}{2}} + 9\,x^{-\frac{1}{2}} - 2\,x^{-1} - 13) \div (3\,x^{\frac{1}{2}} + 2\,x^{-\frac{1}{2}} - 5)$.

21. $(x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$.

22. $(a^{\frac{5}{2}} - a^2b^{\frac{1}{2}} - a^{\frac{1}{2}}b^2 + b^{\frac{5}{2}}) \div (a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}})$.

23. $(6\,x^{\frac{5}{4}} - 7\,x - 19\,x^{\frac{3}{4}} + 2\,x^{\frac{1}{2}} + 8\,x^{\frac{1}{4}}) \div (2\,x^{\frac{3}{4}} - 3\,x^{\frac{1}{2}} - 4\,x^{\frac{1}{4}})$.

Powers of Powers.

(III.) $$(a^m)^n = a^{mn},$$

for all rational values of m and n.

Ex. 1. $$(x^2)^{-3} = x^{2(-3)} = x^{-6} = \frac{1}{x^6}.$$

Ex. 2. $$(1024^{\frac{1}{2}})^{-\frac{3}{5}} = 1024^{-\frac{3}{10}} = \frac{1}{(\sqrt[10]{1024})^3} = \frac{1}{8}.$$

(i.) m and n both negative integers.

Let $m = -m_1$ and $n = -n_1$, so that m_1 and n_1 are positive.

We have

$$(a^m)^n = (a^{-m_1})^{-n_1} = \left(\frac{1}{a^{m_1}}\right)^{-n_1} = (a^{m_1})^{n_1} = a^{m_1n_1} = a^{(-m_1)(-n_1)} = a^{mn}.$$

In a similar manner the principle can be proved for other cases in which the exponents are 0 or negative integers.

(ii.) m a fraction, and n a positive or a negative integer, or 0.

Let $m = \frac{p}{q}$, wherein q is a positive integer and p is a positive or a negative integer.

We then have

$$(a^m)^n = (a^{\frac{p}{q}})^n = [(a^{\frac{1}{q}})^p]^n = (a^{\frac{1}{q}})^{pn} = a^{\frac{pn}{q}} = a^{\frac{p}{q}n} = a^{mn}.$$

In a similar manner the principle can be proved when m is an integer and n is a fraction.

(iii.) m and n both fractions. Let $m = \frac{p}{q}$, and $n = \frac{r}{s}$.

If $(a^{\frac{p}{q}})^{\frac{r}{s}}$ be raised to the qsth, $= sq$th power, we have

$$[(a^{\frac{p}{q}})^{\frac{r}{s}}]^{qs} = \{[(a^{\frac{p}{q}})^{\frac{r}{s}}]^s\}^q = [(a^{\frac{p}{q}})^r]^q = [(a^{\frac{p}{q}})^q]^r = (a^p)^r = a^{pr}.$$

Consequently $(a^{\frac{p}{q}})^{\frac{r}{s}}$ is the qs root of a^{pr}; or, by definition of a fractional power,

$$(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}} = a^{\frac{p}{q}\cdot\frac{r}{s}}.$$

EXERCISES V.

Simplify each of the following expressions:

1. $(x^2)^{-2}$.	2. $(a^3)^{\frac{1}{3}}$.	3. $[(-x)^{\frac{1}{2}}]^2$.	4. $(x^{-3})^4$.
5. $(x^{-\frac{2}{5}})^{15}$.	6. $(a^{-3})^{\frac{1}{6}}$.	7. $(b^3)^{-\frac{4}{9}}$.	8. $(x^{-2})^{-5}$.
9. $(x^{-\frac{1}{3}})^{-\frac{1}{2}}$.	10. $(a^n)^{-2}$.	11. $(a^{-m})^{-3}$.	12. $(a^{-\frac{p}{q}})^{-\frac{m}{n}}$.
13. $(\sqrt[3]{a^{-2}})^4$.	14. $(\sqrt{a})^{-\frac{3}{2}}$.	15. $(\sqrt[5]{x^{\frac{4}{3}}})^{-\frac{1}{4}}$.	16. $(\sqrt[q]{a^{-m}})^{-3}$.

Powers of Products.

(IV.) $(ab)^m = a^m b^m$, for all rational values of m.

Ex. 1. $(2\,x)^{-3} = 2^{-3}x^{-3} = \dfrac{1}{8\,x^3}.$

Ex. 2. $(3\,x^{-\frac{1}{2}}y^2)^{-4} = 3^{-4}x^2y^{-8} = \dfrac{x^2}{81\,y^8}.$

(i.) m a negative integer. Let $m = -m_1$, so that m_1 is positive.

Then $(ab)^m = (ab)^{-m_1} = \dfrac{1}{(ab)^{m_1}} = \dfrac{1}{a^{m_1}b^{m_1}} = a^{-m_1}b^{-m_1} = a^m b^m.$

(ii.) m a fraction. Let $m = \dfrac{p}{q}$, where p is a positive or negative integer, and q is a positive integer.

If $(ab)^{\frac{p}{q}}$ be raised to the qth power, we have

$$\begin{aligned}[(ab)^{\frac{p}{q}}]^q &= (ab)^p, \text{ since } q \text{ is an integer,}\\ &= a^pb^p, \text{ by (i.).}\end{aligned}$$

But $$(a^{\frac{p}{q}}b^{\frac{p}{q}})^q = (a^{\frac{p}{q}})^q(b^{\frac{p}{q}})^q = a^pb^p.$$

Therefore $$[(ab)^{\frac{p}{q}}]^q = (a^{\frac{p}{q}}b^{\frac{p}{q}})^q;\ \text{whence } (ab)^{\frac{p}{q}} = a^{\frac{p}{q}}b^{\frac{p}{q}}.$$

Powers of Quotients.

(V.) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$, for all rational values of m.

Ex. 1. $\left(\dfrac{a^{\frac{1}{2}}}{b^3}\right)^{-3} = \dfrac{a^{-\frac{3}{2}}}{b^{-9}} = \dfrac{b^9}{a^{\frac{3}{2}}}.$ Ex. 2. $\left(\dfrac{4^{-3}}{x^2y^{-1}}\right)^{-\frac{1}{2}} = \dfrac{4^{\frac{3}{2}}}{x^{-1}y^{\frac{1}{2}}} = \dfrac{8\,x}{y^{\frac{1}{2}}}.$

We have $$\left(\frac{a}{b}\right)^m = (ab^{-1})^m = a^mb^{-m} = \frac{a^m}{b^m}.$$

EXERCISES VI.

Simplify each of the following expressions:

1. $(a^{\frac{1}{2}}x^{-1})^{-2}.$ 2. $(\frac{1}{4}\,a)^{-\frac{1}{2}}.$ 3. $(8\,a^{-6})^{\frac{1}{3}}.$

4. $(a^{-1}b^{-3})^{-4}.$ 5. $(2\,a^{\frac{3}{2}}x)^{\frac{5}{6}}.$ 6. $(x^{\frac{1}{3}}a^{-\frac{1}{2}})^{-12}.$

7. $\left(\dfrac{x^{\frac{2}{3}}}{y^{-\frac{1}{4}}}\right)^{-6}$. 8. $\left(\dfrac{-2^3 a^{-3}}{4 b^3}\right)^{-2}$. 9. $\left(\dfrac{4 x^{-\frac{1}{2}}}{y^5}\right)^{-\frac{1}{5}}$.

10. $\left(\dfrac{8 a^2}{27 a^{-3} y^{\frac{1}{3}}}\right)^{-\frac{1}{3}}$. 11. $\left(\dfrac{2 x^{\frac{4}{5}}}{3 a^{-2} b^2}\right)^{-5}$. 12. $\left(\dfrac{5 a^{-\frac{1}{6}} b^{\frac{1}{3}}}{6 x^{-2}}\right)^3$.

13. $\left(\dfrac{\sqrt{a}}{\sqrt[3]{x^2}}\right)^{-6}$. 14. $\left(\dfrac{2\sqrt[3]{a^{-2}}}{3\sqrt{b^{-3}}}\right)^6$. 15. $\left(\dfrac{3\sqrt[4]{x^3}}{5\sqrt{a^{-3}}}\right)^2$.

EXERCISES VII.

MISCELLANEOUS EXAMPLES.

Simplify each of the following expressions:

1. $\dfrac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}-b^{\frac{3}{2}}}{a-b}$. 2. $\dfrac{a-x}{a^{\frac{1}{3}}-x^{\frac{1}{3}}}-\dfrac{a+x}{a^{\frac{1}{3}}+x^{\frac{1}{3}}}$.

3. $\dfrac{a^{\frac{1}{2}}x^{\frac{1}{4}}+a^{\frac{1}{4}}x^{\frac{1}{2}}}{a^{\frac{1}{2}}+x^{\frac{1}{2}}}\cdot\dfrac{a-x}{a^{\frac{1}{4}}+x^{\frac{1}{4}}}$. 4. $\dfrac{x^{\frac{1}{2}}+1}{x+x^{\frac{1}{2}}+1}\div\dfrac{1}{x^{1.5}-1}$.

5. $\dfrac{1}{a^{\frac{1}{4}}+a^{\frac{1}{8}}+1}+\dfrac{1}{a^{\frac{1}{4}}-a^{\frac{1}{8}}+1}-\dfrac{2 a^{\frac{1}{4}}}{a^{\frac{1}{2}}-a^{\frac{1}{4}}+1}$.

Find the square root of each of the following expressions:

6. $x^{\frac{1}{2}}+x^{-\frac{1}{2}}+2$. 7. $a^{-4}x+2 a^{-\frac{3}{2}}x^{-\frac{3}{2}}+ax^{-4}$.

8. $4 x^{-4}-12 x^{-3}+13 x^{-2}-6 x^{-1}+1$.

9. $9 x^2+10 x^{-2}-4 x^{-4}+x^{-6}-12$.

10. $a^2-\frac{3}{2} a^{\frac{3}{2}}-\frac{3}{2} a^{\frac{1}{2}}+\frac{41}{16} a+1$.

11. $\frac{9}{4} x^3-5 x^{\frac{5}{2}}y^{\frac{1}{2}}+\frac{179}{45} x^2y-\frac{4}{3} x^{\frac{3}{2}}y^{\frac{3}{2}}+\frac{4}{25} xy^2$.

Find the cube root of each of the following expressions:

12. $x^{-6}-6 x^{-5}+12 x^{-4}-8 x^{-3}$. 13. $8 x-36 x^{\frac{7}{6}}-27 x^{\frac{3}{2}}+54 x^{\frac{4}{3}}$.

14. $x^{\frac{3}{2}}-3 x^{\frac{4}{3}}+3 x^{\frac{7}{6}}+2 x+3 x^{\frac{5}{4}}-3 x^{\frac{5}{6}}-6 x^{\frac{13}{12}}+3 x^{\frac{11}{12}}+x^{\frac{3}{4}}$.

CHAPTER XVIII.

QUADRATIC EQUATIONS.

1. A **Quadratic Equation** is an equation of the second degree in the unknown number or numbers.

E.g., $x^2 = 25$, $x^2 - 5x + 6 = 0$, $x^2 + 2xy = 7$.

A **Complete Quadratic Equation**, in one unknown number, is one which contains a term (or terms) in x^2, a term (or terms) in x, and a term (or terms) free from x, as $x^2 - 2ax + b = cx - d$.

A **Pure Quadratic Equation** is an incomplete quadratic equation which has no term in x, as $x^2 - 9 = 0$.

Pure Quadratic Equations.

2. Ex. 1. Solve the equation $6x^2 - 7 = 3x^2 + 5$.

Transferring $3x^2$ to the first member, and 7 to the second member,

$$6x^2 - 3x^2 = 5 + 7,$$

or

$$3x^2 = 12.$$

Dividing by 3,

$$x^2 = 4.$$

The value of x is a number whose square is 4. But

$$2^2 = 4, \text{ and } (-2)^2 = 4.$$

Therefore

$$x = \pm 2.$$

3. This example illustrates the following principle of equivalent equations:

The positive square root of the first member of an equation may be equated in turn to the positive and to the negative square root of the second member.

Ex. 2. Solve the equation $(x-2)(x+2)=-6$.

Simplifying, $x^2-4=-6$.

Transferring -4, $x^2=-2$.

Equating square roots, $x=\pm\sqrt{-2}$.

These results are imaginary. Yet they satisfy the given equation, since

$$(\pm\sqrt{-2}-2)(\pm\sqrt{-2}+2)=(\pm\sqrt{-2})^2-4=-2-4=-6.$$

In such a case the equation is said to have *imaginary roots*. The meaning of an imaginary result, when it arises in connection with a problem, will be explained in Art. 16.

4. The methods used in Ch. VIII. for solving fractional equations which lead to linear equations apply also to fractional equations which lead to quadratic equations.

Ex. 3. Solve the equation $\dfrac{a+x}{b+x}+\dfrac{x-a}{x-b}=0$.

Clearing of fractions,

$$(a+x)(x-b)+(x-a)(b+x)=0,$$

or, $$x^2+ax-bx-ab+x^2-ax+bx-ab=0.$$

Transferring and uniting terms,

$$2x^2=2ab.$$

Dividing by 2 and equating square roots,

$$x=\pm\sqrt{(ab)}.$$

This equation therefore has *irrational roots*.

EXERCISES I.

Solve each of the following equations:

1. $x^2=729$. **2.** $x^2-25=144$. **3.** $5x^2-27=2x^2$.

4. $\dfrac{3}{x}=\dfrac{x}{27}$. **5.** $\dfrac{8x}{81}=\dfrac{9}{2x}$. **6.** $\dfrac{x^2-1}{4}=2$.

7. $\dfrac{5x^2+12}{8}=4$. **8.** $\dfrac{1}{x^2+1}=\dfrac{1}{5}$. **9.** $\dfrac{18}{x^2-1}=6$.

10. $7x^2-8=9x^2-10$. **11.** $5+16x^2=11x^2+15$.

12. $5x^2+9+7x^2=8x^2+25.$

13. $5(3x^2+1)+81=7(5x^2-16)+18.$

14. $\frac{5}{2x^2}-\frac{4}{3x^2}=\frac{7}{12}.$

15. $\frac{2-x^2}{5}-\frac{7x^2+9}{6}=-\frac{37}{15}.$

16. $7-\frac{15-x}{x^2}=6+\frac{x+10}{x^2}.$

17. $\frac{11}{x^2}+5=7\left(1-\frac{1}{x^2}\right).$

18. $(7+2x)(7-2x)=13.$

19. $(x+\frac{1}{3})(x-\frac{1}{3})=11.$

20. $(x-8)(x+5)=3(3-x).$

21. $(x+2)(x+3)=5(x+1).$

22. $(x+3)^2=49.$

23. $(3x+4)^2-49=576.$

24. $64x^2-80x+25=9.$

25. $(5x+4)^2+(4x-5)^2=82.$

26. $\frac{x+5}{5x+1}=\frac{5x+1}{x+5}.$

27. $\frac{2x-3}{3x-2}=\frac{3x-2}{2x-3}.$

28. $\frac{x+5}{x+13}=\frac{2x+7}{3x+18}.$

29. $\frac{3x-4}{4x+1}=\frac{7x-24}{8x-19}.$

30. $\frac{x+3}{8}-\frac{10}{x+1}=\frac{1}{2}.$

31. $\frac{6x}{7}-\frac{14+x^2}{2x+7}=3.$

32. $(2x-3)(3x-4)-(x-13)(x-4)=40.$

33. $(5x-7)(3x+8)-(x-10)(9-x)=1634.$

34. $\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{3x^2-7}{x^2-1}.$

35. $\frac{5}{x-5}-\frac{3}{2x+3}=\frac{132}{77(x-6)}.$

36. $\frac{64}{x+7}+\frac{11}{x-8}+\frac{6}{x+2}=\frac{81}{x+12}.$

37. $(5-x)(3-x)(1+x)+(5+x)(3+x)(1-x)=16.$

38. $ax^2=b^4.$

39. $(a-bx)^2=c^2.$

40. $ax^2+b^2=bx^2+a^2.$

41. $(x+a)(x-a)=3a^2.$

42. $m^2x^2-4mx+4=9.$

43. $ax^2+\frac{b}{a}=bx^2+\frac{a}{b}.$

44. $\frac{a}{a+x}+\frac{b}{b+x}=1.$

45. $\frac{a^2}{a^2+x^2}=\frac{b^2}{x^2-a^2+b^2}.$

46. $\frac{ax-b}{a-bx}=\frac{bx+a}{b+ax}.$

47. $\frac{x+1}{x-1}=\frac{a+bx+cx^2}{a-bx+cx^2}.$

Solution by Factoring.

5. The principle on which the solution of an equation by factoring depends was proved in Ch. VI, Art. 43. The methods given in Ch. VI, Arts. 9–13; Ch. XV, Art. 33; and Ch. XVI, Art. 20, enable us to factor any quadratic expression. The roots of the given quadratic equation are the roots of the equations obtained by equating to 0 each of its factors.

Ex. 1. Solve the equation $3x^2 + 5x - 2 = 0$.

Dividing by 3, $$x^2 + \tfrac{5}{3}x - \tfrac{2}{3} = 0.$$

Adding and subtracting $(\frac{5}{2\times 3})^2, = \frac{25}{36}$, we have

$$x^2 + \tfrac{5}{3}x + \tfrac{25}{36} - \tfrac{25}{36} - \tfrac{2}{3} = 0.$$

or, $$(x + \tfrac{5}{6})^2 - \tfrac{49}{36} = 0.$$

Factoring, $$(x + \tfrac{5}{6} + \tfrac{7}{6})(x + \tfrac{5}{6} - \tfrac{7}{6}) = 0,$$

or, $$(x + 2)(x - \tfrac{1}{3}) = 0,$$

Equating each factor to 0,

$$x + 2 = 0, \text{ whence } x = -2;$$

$$x - \tfrac{1}{3} = 0, \text{ whence } x = \tfrac{1}{3}.$$

Ex. 2. Solve the equation $2x^2 + 2x - 1 = 0$.

Dividing by 2, $$x^2 + x - \tfrac{1}{2} = 0.$$

Adding and subtracting $(\frac{1}{2})^2, = \frac{1}{4}$,

$$x^2 + x + \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{2} = 0,$$

or $$(x + \tfrac{1}{2})^2 - (\tfrac{1}{2}\sqrt{3})^2 = 0.$$

Factoring, $(x + \frac{1}{2} + \frac{1}{2}\sqrt{3})(x + \frac{1}{2} - \frac{1}{2}\sqrt{3}) = 0$.

Equating factors to 0,

$$x + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3} = 0,\ x + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3} = 0.$$

Whence $x = -\frac{1}{2} - \frac{1}{2}\sqrt{3}$, and $-\frac{1}{2} + \frac{1}{2}\sqrt{3}$.

Such roots are usually written $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$.

Ex. 3. Solve the equation $x^2 - 2x + 19 = 0$.

Adding and subtracting $(-1)^2, = 1$,

$$x^2 - 2x + 1 - 1 + 19 = 0,$$

or, since $-1 + 19 = 18 = -(-18) = -(\sqrt{-18})^2,$
$$= -(3\sqrt{-2})^2,$$

$$(x-1)^2 - (3\sqrt{-2})^2 = 0.$$

Factoring, $(x - 1 + 3\sqrt{-2})(x - 1 - 3\sqrt{-2}) = 0.$

Equating factors to 0,

$$x - 1 + 3\sqrt{-2} = 0,\ x - 1 - 3\sqrt{-2} = 0.$$

Whence, $x = 1 \pm 3\sqrt{-2}.$

EXERCISES II.

Solve each of the following equations:

1. $x^2 - 6x + 5 = 0.$
2. $x^2 - 7x + 10 = 0.$
3. $x^2 - 4x - 21 = 0.$
4. $x^2 = 11x + 12.$
5. $3x^2 + 4x + 1 = 0.$
6. $9x^2 - 12x + 4 = 0.$
7. $6x^2 + 13x - 8 = 0.$
8. $11x^2 - 7x - 18 = 0.$
9. $7x^2 - 20x + 8 = 0.$
10. $7 - 12x^2 = 17x.$
11. $20x^2 - 79x + 77 = 0.$
12. $8x^2 + 13x - 82 = 0.$
13. $x^2 - 2x - 1 = 0.$
14. $x^2 - 6x - 71 = 0.$
15. $x^2 - 2x + 2 = 0.$
16. $x^2 - 4x + 13 = 0.$
17. $(x + 8)(x + 3) = x - 6.$
18. $(x + 7)(x - 7) = 2(x + 50).$
19. $(2x + 1)(x + 2) = 3x^2 - 4.$
20. $(x - 1)(2x + 3) = 4x^2 - 22.$
21. $x^2 - 3 = \frac{1}{6}(x - 3).$
22. $x(x + 5) = 5(40 - x) + 27.$
23. $\dfrac{x}{x + 120} = \dfrac{14}{3x - 10}.$
24. $\dfrac{x + 7}{2x + 3} = \dfrac{3x - 5}{x + 3}.$
25. $\dfrac{x + 3}{4} - \dfrac{5}{x - 6} = \dfrac{x + 11}{6}.$
26. $\dfrac{5}{x} + \dfrac{4x + 7}{x + 1} = -\dfrac{3}{2}.$
27. $\dfrac{3}{x - 1} + \dfrac{5}{x - 2} = \dfrac{6}{x - 3}.$
28. $\dfrac{x + 2}{x + 3} - \dfrac{x + 4}{x + 5} = -\dfrac{14}{x + 3}.$

29. $\dfrac{9x+1}{9x-3x^2}=\dfrac{x}{21-7x}-\dfrac{x+3}{21x}$.

30. $\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{x-3}=0$.

31. $\dfrac{5x-1}{x+3}+\dfrac{7x^2-106}{8x^2-72}=-\dfrac{1}{8}$.

32. $\dfrac{x-2}{x+2}+\dfrac{x+2}{x-2}=\dfrac{2(x+3)}{x-3}$.

33. $\dfrac{x+24}{5x^2-5}=\dfrac{x-7}{x+1}-\dfrac{1}{2x-2}$.

34. $\dfrac{4x+67}{40x^2-36}+\dfrac{x}{30x^2-27}=\dfrac{2}{3}$.

35. $x^2+11ax+28a^2=0$.

36. $x^2-14mx+33m^2=0$.

37. $x^2-2ax+a^2-b^2=0$.

38. $x^2-3ax+2a^2-ab-b^2=0$.

39. $x^2-(2m-1)x+m^2-m-6=0$.

40. $x^2-(3a+2b)x+6ab=0$.

41. $ax^2+(a+2)x+2=0$.

42. $bx^2-2(b+c)x+4c=0$.

43. $(a+1)x^2-ax-1=0$.

44. $(a^2+3a-10)x^2-(2a+3)x+1=0$.

45. $x^2-2(a+b)x+(a+b+c)(a+b-c)=0$.

46. $(m-n)x^2-(m+n)x+2n=0$.

47. $\dfrac{a}{x-b}+\dfrac{b}{x-a}=2$.

48. $\dfrac{x-4a}{2a-b}+\dfrac{2a+b}{x}=0$.

49. $\dfrac{a}{x}+\dfrac{x-a}{ab(b-1)}=\dfrac{2}{b}$.

50. $\dfrac{an}{x+4n}-\dfrac{an}{x-4n}=2$.

Solution by Completing the Square.

6. The following examples illustrate the solution of a quadratic equation by the method called *Completing the Square*.

Ex. 1. Solve the equation $x^2-5x+6=0$.

Transferring 6, $x^2-5x=-6$.

To complete the square in the first member, we add $(-\frac{5}{2})^2$, $=\frac{25}{4}$, to this member, and therefore also to the second. We then have

$$x^2-5x+\tfrac{25}{4}=\tfrac{25}{4}-6=\tfrac{1}{4}.$$

Equating square roots, $x-\frac{5}{2}=\pm\frac{1}{2}$, by Art. 2.

Whence, $x = \frac{5}{2} \pm \frac{1}{2}.$

Therefore the required roots are 3 and 2.

Ex. 2. Solve the equation

$$7x^2 + 5x + 1 = 0.$$

Transferring 1, $7x^2 + 5x = -1.$

Dividing by 7, $x^2 + \frac{5}{7}x = -\frac{1}{7}.$

Adding $\left(\frac{5}{2 \times 7}\right)^2 = \frac{25}{196}$, $x^2 + \frac{5}{7}x + \frac{25}{196} = \frac{25}{196} - \frac{1}{7} = \frac{-3}{196}.$

Equating square roots, $x + \frac{5}{14} = \pm \frac{1}{14}\sqrt{-3}.$

Whence, $x = -\frac{5}{14} \pm \frac{1}{14}\sqrt{-3}.$

Therefore the required roots are

$$-\frac{5}{14} + \frac{1}{14}\sqrt{-3} \text{ and } -\frac{5}{14} - \frac{1}{14}\sqrt{-3}.$$

Ex. 3. Solve the equation

$$(a^2 - b^2)x^2 - 2a^2x + a^2 = 0.$$

Transferring a^2, $(a^2 - b^2)x^2 - 2a^2x = -a^2.$

Dividing by $a^2 - b^2$, $x^2 - \frac{2a^2x}{a^2 - b^2} = \frac{-a^2}{a^2 - b^2}.$

Adding $\left(-\frac{a^2}{a^2 - b^2}\right)^2, = \frac{a^4}{(a^2 - b^2)^2}$, to both members,

$$x^2 - \frac{2a^2x}{a^2 - b^2} + \frac{a^4}{(a^2 - b^2)^2} = -\frac{a^2}{a^2 - b^2} + \frac{a^4}{(a^2 - b^2)^2} = \frac{a^2b^2}{(a^2 - b^2)^2}.$$

Equating square roots, $x - \frac{a^2}{a^2 - b^2} = \pm \frac{ab}{a^2 - b^2}.$

Whence, $x = \frac{a^2 \pm ab}{a^2 - b^2}.$

Therefore the required roots are $\frac{a}{a - b}$ and $\frac{a}{a + b}.$

The preceding examples illustrate the following method of procedure:

Bring the terms in x and x^2 to the first member, and the terms free from x to the second member, uniting like terms.

If the resulting coefficient of x^2 be not $+1$, divide both members by this coefficient.

Complete the square by adding to both members the square of half the coefficient of x.

Equate the positive square root of the first member to the positive and negative square roots of the second member.

Solve the resulting equations.

EXERCISES III.

Solve each of the following equations:

1. $x^2 - 4x + 3 = 0.$ **2.** $x^2 - 5x = -4.$

3. $x^2 + 2x + 1 = 0.$ **4.** $2x^2 - 7x + 3 = 0.$

5. $3x^2 - 53x + 34 = 0.$ **6.** $14x - 49x^2 - 1 = 0.$

7. $x^2 - 4x + 7 = 0.$ **8.** $110x^2 - 21x + 1 = 0.$

9. $x^2 - 2x + 6 = 0.$ **10.** $x^2 - 1 + x(x-1) = x^2.$

11. $(3x-2)(x-1) = 14.$ **12.** $(2x-1)(x-2) = (x+1)^2.$

13. $x + \frac{1}{x} = 5\frac{1}{5}.$ **14.** $x - \frac{1}{x} = 1\frac{1}{2}.$

15. $x - 1 = \frac{12}{x}.$ **16.** $\frac{21}{x} = x - 4.$

17. $\frac{1}{2x} + \frac{1}{3x} = x - \frac{1}{6}.$ **18.** $x + \frac{1}{x} = 7 + \frac{1}{7}.$

19. $\frac{7}{x-4} = x + 2.$ **20.** $2x + 5 = \frac{11}{4x - 11}.$

21. $\frac{x+3}{x+9} = -\frac{x-4}{x-1}.$ **22.** $\frac{x+1}{x+5} = \frac{3x+1}{7x-1}.$

23. $\frac{10}{1-x} + \frac{27}{1-2x} = 5.$ **24.** $\frac{x+3}{x-5} - \frac{2x-4}{x+5} = 2.$

25. $(2x-3)^2 = 8x.$ **26.** $(2x+1)(x+2) = 3x^2 - 4.$

27. $(5x-3)^2 - 7 = 40x - 47.$ **28.** $(x+1)(2x+3) = 4x^2 - 22.$

29. $(x-7)(x-4) + (2x-3)(x-5) = 103.$

30. $10(2x+3)(x-3) + (7x+3)^2 = 20(x+3)(x-1).$

31. $(x-1)(x-3) + (x-3)(x-5) = 32.$

32. $(x-1)(x-2) + (x-3)(x-4) = (x-1)^2 - 2.$

33. $\dfrac{6}{x-5}-\dfrac{3}{x-4}=\dfrac{8}{x-3}.$

34. $\dfrac{12}{x+1}-\dfrac{7}{6-x}=-\dfrac{15}{x-2}.$

35. $\dfrac{5x}{x+2}+\dfrac{6}{x+3}+\dfrac{7}{x+4}=5.$

36. $\dfrac{2x-7}{2x-1}-\dfrac{7}{5x-4}+\dfrac{11}{3x-4}=1.$

37. $\dfrac{3x}{x^2+3x+2}+\dfrac{6}{x^2+5x+6}=\dfrac{8}{x^2+4x+3}.$

38. $\dfrac{\frac{1}{6}x}{x^2-9x+20}-\dfrac{1}{x^2-7x+10}=\dfrac{2}{x^2-6x+8}.$

39. $\dfrac{x+2}{6x^2+5x+1}+\dfrac{1+x}{10x^2+7x+1}=\dfrac{1-3x}{15x^2+8x+1}.$

40. $x-\dfrac{a}{b}=\dfrac{b}{a}-\dfrac{1}{x}.$

41. $\dfrac{n+x}{n-x}+\dfrac{n-x}{n+x}=\dfrac{n^2}{n^2-x^2}.$

42. $x=\dfrac{3}{(a-b)^2x}-\dfrac{2}{a-b}.$

43. $\dfrac{x^2+1}{n^2x-2n}-\dfrac{1}{2-nx}=\dfrac{x}{n}.$

44. $\dfrac{a-2b}{8x^2-2b^2}=\dfrac{1}{2x+b}-\dfrac{1}{2a}.$

45. $\dfrac{a}{nx-x}-\dfrac{a-1}{x^2-2nx^2+n^2x^2}=1.$

46. $\dfrac{x-a+b}{x+a-b}=\dfrac{a-b-x}{a+b+x}.$

47. $\dfrac{ax}{ax+1}=\dfrac{1-a}{a^2x^2-a-a^2x+ax}.$

48. $\left(\dfrac{a+x}{a-x}\right)^2+\dfrac{7}{2}\cdot\dfrac{a+x}{a-x}+3=0.$

General Solution

7. The most general form of the quadratic equation in one unknown number is evidently

$$ax^2+bx+c=0.$$

The coefficient a is assumed to be *positive* and not 0, but b and c may either or both be positive or negative, or 0.

Dividing by a, $$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Transferring $\frac{c}{a}$, $$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Adding $\left(\frac{b}{2a}\right)^2, =\frac{b^2}{4a^2}$, $$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$
$$=\frac{b^2-4ac}{4a^2}.$$

Equating square roots, $$x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a}.$$

Whence, $$x=-\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a},$$

and $$x=-\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}.$$

8. The roots of any quadratic equation can be obtained by substituting in the general solution the particular values of the coefficients a, b, and c.

Ex. Solve the equation $3x^2+7x-10=0$.

We have $$a=3,\ b=7,\ c=-10.$$

Substituting these values in the general solution, we obtain
$$x=-\tfrac{7}{6}+\tfrac{1}{6}\sqrt{[49-4\times 3(-10)]}=1,$$
and $$x=-\tfrac{7}{6}-\tfrac{1}{6}\sqrt{[49-4\times 3(-10)]}=-\tfrac{10}{3}.$$

EXERCISES IV.

Solve each of the following equations:

1. $2x^2=3x+2$.
2. $5x^2-6x+1=0$.
3. $9x(x+1)=28$.
4. $x^2-b^2=2ax-a^2$.
5. $x^2+6ax+1=0$.
6. $x^2+1=2\tfrac{1}{6}x$.
7. $(x-5)^2+(x-10)^2=37$.
8. $2x(3n-4x)=n^2$.
9. $n^2(x^2+1)=a^2+2n^2x$.
10. $x^2+(x+a)^2=a^2$.

Relation between Roots and Coefficients.

9. If the roots of the quadratic equation

$$ax^2 + bx + c = 0, \text{ or } x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

be designated by r_1 and r_2, we have

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a},$$

$$r_2 = -\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}.$$

The sum of the roots is

$$r_1 + r_2 = -\frac{b}{a}. \tag{1}$$

The product of the roots is

$$r_1 r_2 = \left[-\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a}\right] \times \left[-\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}\right]$$

$$= \left[-\frac{b}{2a}\right]^2 - \left[\frac{\sqrt{(b^2 - 4ac)}}{2a}\right]^2 = \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{c}{a}. \tag{2}$$

The relations (1) and (2) may be expressed thus:

(i.) *If the coefficient of the second power of the unknown number be* 1, *the sum of the roots is equal to the coefficient of the first power of the unknown number, with sign reversed.*

(ii.) *If the coefficient of the second power of the unknown number be* 1, *the product of the roots is equal to the term free from the unknown number.*

E.g., the roots of the equation $x^2 - 5x + 6 = 0$ are 2 and 3; their sum is 5 (the coefficient of x with sign reversed), and their product is 6 (the term free from x).

The roots of the equation $6x^2 - x - 2 = 0$, or $x^2 - \frac{1}{6}x - \frac{1}{3} = 0$, are $\frac{2}{3}$ and $-\frac{1}{2}$; their sum is $\frac{1}{6}$, and the product is $-\frac{1}{3}$.

10. Formation of an Equation from its Roots. — The relations of the last article enable us to form an equation if its roots be given. We may always assume that the coefficient of the second power of the unknown number is 1.

Ex. 1. Form the equation whose roots are -1, 2.

We have $r_1 + r_2 = -1 + 2 = 1$, the coefficient of x, with sign reversed; and $r_1 r_2 = -1 \times 2 = -2$, the term free from x.

Therefore the required equation is $x^2 - x - 2 = 0$.

Ex. 2. Form the equation whose roots are $1 + 2\sqrt{3}$, $1 - 2\sqrt{3}$.

We have $r_1 + r_2 = (1 + 2\sqrt{3}) + (1 - 2\sqrt{3}) = 2$;

and $\qquad r_1 r_2 = (1 + 2\sqrt{3})(1 - 2\sqrt{3}) = 1 - 12 = -11.$

Therefore the required equation is $x^2 - 2x - 11 = 0$.

11. It follows from Art. 9, that the quadratic equation may be written in the form

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0,$$

or

$$(x - r_1)(x - r_2) = 0.$$

Ex. Form the equation whose roots are -1, 2.

We have $(x + 1)(x - 2) = 0$, or $x^2 - x - 2 = 0$.

When the roots are irrational or imaginary, the method of the preceding article is to be preferred.

EXERCISES V.

Form the equations whose roots are:

1. $8, 2$. **2.** $-5, -3$. **3.** $10, 10$. **4.** $7, -3$.

5. $4, -10$. **6.** $2\frac{1}{2}, 1\frac{3}{5}$. **7.** $-\frac{2}{3}, -1\frac{1}{2}$. **8.** $-\frac{1}{4}, 8$.

9. $2, 0$. **10.** a, b. **11.** $-a, -1$. **12.** $a^2, -4a^2$.

13. $\sqrt{2}, -\sqrt{2}$. **14.** $\frac{1}{2}\sqrt{-3}, -\frac{1}{2}\sqrt{-3}$.

15. $1 + \sqrt{7}, 1 - \sqrt{7}$. **16.** $\frac{1}{2} - \frac{1}{3}\sqrt{11}, \frac{1}{2} + \frac{1}{3}\sqrt{11}$.

17. $3 - \sqrt{-5}, 3 + \sqrt{-5}$. **18.** $\frac{2}{3} - \frac{1}{2}\sqrt{-1}, \frac{2}{3} + \frac{1}{2}\sqrt{-1}$.

Nature of the Roots.

12. In many applications it is important to know, without having to solve an equation, the nature of its roots, *i.e.*, whether they are both *real and unequal*, whether they are both *real and equal*, whether they are *imaginary*.

In the general solution

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a}, \quad r_2 = -\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a},$$

of the equation $\qquad ax^2 + bx + c = 0,$

a, b, and c are limited to real, rational values.

(i.) *The two roots are real and unequal when* $b^2 - 4ac$ *is positive*, i.e., *when* $b^2 - 4ac > 0$.

E.g., in $\qquad x^2 + 4x - 12 = 0,$

$$a = 1,\ b = 4,\ c = -12;\ \text{and since } b^2 - 4ac,\ = 16 + 48,$$

is *positive*, the roots of this equation are real and unequal.

(ii.) *The two roots are real and equal when* $b^2 - 4ac$ *is equal to* 0; i.e., *when* $b^2 = 4ac$.

E.g., in $\qquad x^2 - 4x + 4 = 0,$

$$a = 1,\ b = -4,\ c = 4;\ \text{and since } b^2 = 4ac,$$

the roots of this equation are real and equal.

(iii.) *The two roots are conjugate complex numbers when* $b^2 - 4ac$ *is negative*; i.e., *when* $b^2 - 4ac < 0$.

E.g., in $\qquad x^2 - 2x + 3 = 0,$

$$a = 1,\ b = -2,\ c = 3;\ \text{and since } b^2 - 4ac,\ = 4 - 12,\ = -8,$$

is *negative*, the roots of this equation are complex numbers.

EXERCISES VI.

Without solving the following equations, determine the nature of the roots of each one:

1. $x^2 + 17x + 70 = 0$. **2.** $x^2 + 12x = -40$. **3.** $x^2 + 5x - 14 = 0$.

4. $x^2 - x = 12$. **5.** $x^2 - 8x + 25 = 0$.

6. $x^2 - 8x = 16$. **7.** $9x^2 - 12x + 4 = 0$.

8. $8x^2 - 2x - 25 = 0$. **9.** $16x^2 + 8x + 49 = 0$.

10. $10x^2 - 21x - 10 = 0$.

For what values of m are the roots of each of the following equations equal? For what values of m are the roots real and unequal? And for what values of m are the roots complex numbers?

11. $mx^2+4x+1=0$. **12.** $2x^2+mx+1=0$.

13. $3x^2+6x+m=0$. **14.** $mx^2+mx+1=0$.

IRRATIONAL EQUATIONS.

13. An irrational equation may lead to a quadratic equation when rationalized.

Ex. 1. Solve the equation $x+\sqrt{(25-x^2)}=7$.

Transferring x, $\sqrt{(25-x^2)}=7-x$. (1)

Squaring, $25-x^2=49-14x+x^2$. (2)

The roots of this equation are 3, 4.

Both roots of (2) satisfy the given equation, since

$$3+\sqrt{(25-9)}=7, \text{ and } 4+\sqrt{(25-16)}=7.$$

Ex. 2. Solve the equation $x-\sqrt{(25-x^2)}=1$.

Transferring x, $-\sqrt{(25-x^2)}=1-x$. (1)

Squaring, $25-x^2=1-2x+x^2$. (2)

The roots of this equation are 4 and -3.

The number 4 is a root of the given equation, since

$$4-\sqrt{(25-16)}=1;$$

but the number -3 is not a root of the given equation, since

$$-3-\sqrt{(25-9)}=-7, \text{ not } 1.$$

Therefore the root -3 was introduced by squaring. Now observe that the same rational equation (2) would have been obtained, if the given equation had been

$$x+\sqrt{(25-x^2)}=1; \quad (3)$$

that is, if the surd term had been of opposite sign. The root -3 satisfies equation (3), since

$$-3+\sqrt{(25-9)}=-3+4=1.$$

Therefore equation (2) is equivalent to equations (1) and (3) jointly.

It frequently happens that no root can be found to satisfy an equation obtained by giving to the square root either its positive or its negative value.

In Ex. 1, the equation thus derived is

$$x - \sqrt{(25 - x^2)} = 7,$$

and is not satisfied by either of the roots obtained. The equation is then said to be *impossible.*

Ex. 3. Solve the equation

$$\sqrt{(2x + 3)} - \sqrt{(7 - x)} = 1.$$

If both *positive* and *negative* square roots be admitted, the given equation is equivalent to the four equations:

$$\sqrt{(2x+3)} + \sqrt{(7-x)} = 1 \ (1), \quad \sqrt{(2x+3)} - \sqrt{(7-x)} = 1 \ (2),$$
$$-\sqrt{(2x+3)} + \sqrt{(7-x)} = 1 \ (3), \quad -\sqrt{(2x+3)} - \sqrt{(7-x)} = 1 \ (4).$$

The same rational integral equation will evidently be derived by rationalizing any one of these equations.

In (1) transferring $\sqrt{(7 - x)}$,

$$\sqrt{(2x + 3)} = 1 - \sqrt{(7 - x)}.$$

Squaring, $\quad 2x + 3 = 1 - 2\sqrt{(7 - x)} + 7 - x,$

or $\quad 3x - 5 = -2\sqrt{(7 - x)}.$

Again squaring, $\quad 9x^2 - 30x + 25 = 28 - 4x,$

or $\quad 9x^2 - 26x - 3 = 0.$

The roots of this equation are 3 and $-\frac{1}{9}$. By substitution we find that equation (2) is satisfied by the root 3, and equation (3) by the root $-\frac{1}{9}$. The other two equations are impossible.

Consequently, in solving an irrational equation, we must expect to obtain not only its roots, but also the roots of the other equations obtained by changing the signs of the radicals in all possible ways. Some of these equations will be impossible. The roots of the other irrational equations will be the roots of the rational equation.

14. Ex. Solve the equation

$$\sqrt{(3x^2-2x+4)}-3x^2+2x=-16.$$

Since $\qquad -3x^2+2x=-(3x^2-2x+4)+4,$

we may take $\sqrt{(3x^2-2x+4)}$ as the unknown number, replacing it temporarily by y. We then have the quadratic equation

$$y-y^2+4=-16.$$

The roots of this equation are 5, and -4.

Equating $\sqrt{(3x^2-2x+4)}$ to each of these roots, we have

$$\sqrt{(3x^2-2x+4)}=5, \text{ whence } x=3, \ -\tfrac{7}{3}.$$

$$\sqrt{(3x^2-2x+4)}=-4, \text{ whence } x=\tfrac{1}{3}(1\pm\sqrt{37}).$$

The numbers 3, $-\frac{7}{3}$ satisfy the given equation, and are therefore roots of that equation. The numbers $\frac{1}{3}\sqrt{(1\pm\sqrt{37})}$ do not satisfy the given equation.

But if the value of the radical is not restricted to the positive root, the given equation comprises the two equations

$$\sqrt{(3x^2-2x+4)}-3x^2+2x=-16, \tag{1}$$

$$-\sqrt{(3x^2-2x+4)}-3x^2+2x=-16. \tag{2}$$

Then $\frac{1}{3}(1\pm\sqrt{37})$ are roots of (2).

The given equation is said to be in *quadratic form*.

EXERCISES VII.

Solve each of the following equations, and check the results. If a result does not satisfy an equation as written, determine what signs the radical terms must have in order that the result may satisfy the equation.

1. $\sqrt{(x^2-9)}=4.$ **2.** $4x=3\sqrt{(2x^2-4)}.$

3. $3-\sqrt{(3x^2-4x+9)}=0.$ **4.** $5x=2\sqrt{(3x^2-x+15)}.$

5. $\sqrt{[(x-5)-7+\sqrt{(x-12)}]}=0.$

6. $\sqrt{[4x-\sqrt{(2x+3)}]}=3.$

7. $\dfrac{x-1}{\sqrt{x}+1}=4+\dfrac{\sqrt{x}-1}{2}.$ **8.** $\dfrac{x+\sqrt{(x^2+7)}}{28}=\dfrac{1}{\sqrt{(x^2+7)}}.$

9. $\dfrac{2x+\sqrt{(4x^2-1)}}{2x-\sqrt{(4x^2-1)}}=4.$ **10.** $\dfrac{x-\sqrt{(x+1)}}{x+\sqrt{(x+1)}}=\dfrac{5}{11}.$

11. $7\sqrt{x}=3\sqrt{(x^2+3x-59)}$. **12.** $\sqrt{(x+2)}-\sqrt{(x^2+2x)}=0$.

13. $(5-\sqrt{x})^2=2(7+\sqrt{x})$. **14.** $x+5-\sqrt{(x+5)}=6$.

15. $\sqrt{(x-2)}+2\sqrt{(x+3)}-2\sqrt{(3x-2)}=0$.

16. $\sqrt{(2x+9)}+\sqrt{(3x-15)}=\sqrt{(7x+8)}$.

17. $\sqrt{\dfrac{3x-4}{x-5}}+\sqrt{\dfrac{x-5}{3x-4}}=\dfrac{5}{2}$. **18.** $\sqrt{\dfrac{3x+6}{7x-3}}+\sqrt{\dfrac{7x-3}{3x+6}}=\dfrac{13}{6}$.

19. $\dfrac{1}{\sqrt{(x+2)}}+\dfrac{1}{\sqrt{(3x-2)}}=\dfrac{4}{\sqrt{(3x^2+4x-4)}}$.

20. $\dfrac{1}{x-\sqrt{(2-x^2)}}+\dfrac{1}{x+\sqrt{(2-x^2)}}=1$.

21. $x^2-x+2\sqrt{(x^2-x-11)}=14$.

22. $x^2+24=2x+6\sqrt{(2x^2-4x+16)}$.

23. $\sqrt{(2x^2-3x+5)}+2x^2-3x=1$.

24. $\sqrt{(2x^2-7x+7)}+\sqrt{(2x^2+9x-1)}=6$.

25. $\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}=\dfrac{a}{b}$. **26.** $\dfrac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\dfrac{a}{b}$.

27. $\sqrt{(a+x)}+\sqrt{(a-x)}=\dfrac{a}{\sqrt{(a+x)}}$.

28. $\dfrac{x^2}{a-\sqrt{(a^2-x^2)}}-\dfrac{x^2}{a+\sqrt{(a^2-x^2)}}=a$.

29. $\sqrt{(1-x+x^2)}+\sqrt{(1+x+x^2)}=m$.

HIGHER EQUATIONS.

15. Certain equations of higher degree than the second can be solved by means of quadratic equations.

Ex. 1. Solve the equation $x^3-1=0$.

Factoring, $(x-1)(x^2+x+1)=0$.

This equation is equivalent to the two equations

$$x-1=0, \text{ whence } x=1;$$

and $x^2+x+1=0$, whence $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

This example gives the three cube roots of 1, since $x^3-1=0$ is equivalent to

$$x^3=1, \text{ or } x=\sqrt[3]{1}.$$

Therefore the three cube roots of 1 are

$$1, \quad -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, \quad -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}.$$

In general, the three cube roots of any number can be found by multiplying the arithmetical cube root of the number in turn by the three algebraic cube roots of 1.

$$\sqrt[3]{8}=2\sqrt[3]{1}=2, \quad -1\pm\sqrt{-3}.$$

Ex. 2. Solve the equation $x^4-9=2x^2-1$.

Since $x^4=(x^2)^2$, we may take x^2 as the unknown number and solve this equation as a quadratic in x^2.

We then have $\qquad (x^2)^2-2x^2-8,=0.$

Factoring, $\qquad (x^2-4)(x^2+2)=0.$

Whence,

$$x^2-4=0, \text{ or } x=\pm 2; \text{ and } x^2+2=0, \text{ or } x=\pm\sqrt{-2}.$$

In general, any equation containing only two powers of the unknown number, *one of which is the square of the other*, can be solved as a quadratic equation.

Ex. 3. Solve the equation $(x^2-3x+1)^2=6+5(x^2-3x+1)$.

In this example x^2-3x+1 is regarded as the unknown number, and may temporarily be represented by the letter y. The equation then becomes

$$y^2=6+5y; \text{ whence } y=6, \text{ and } -1.$$

We therefore have the two equations

$$x^2-3x+1=6, \text{ whence } x=\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{29};$$

$$x^2-3x+1=-1, \text{ whence } x=2, \ x=1.$$

Therefore the roots of the given equation are $\frac{3}{2}\pm\frac{1}{2}\sqrt{29}, 2, 1$.

Attention is called to the fact that, in each example, we have obtained as many roots as there are units in the degree of the *equation.*

EXERCISES VIII.

Solve each of the following equations:

1. $x^3+1=0$. 2. $x^4-1=0$. 3. $x^6+1=0$.
4. $x^6-1=0$. 5. $(x-1)^3=8$. 6. $x^3=(2a-x)^3$.
7. $(x+1)^4=16$. 8. $x^4+9=10x^2$. 9. $x^4-6x^2=-1$.
10. $x^6-65x^3=-64$. 11. $x^8+5x^4=6$.
12. $(x^2-x+1)^2=3x(x-1)+1$.
13. $(3x^2-5x+1)^2-9x^2+15x=7$.
14. $15x^2-35x-3(7x-3x^2+8)^2+310=0$.
15. $\dfrac{(a+x)^4+(a-x)^4}{(a+x)^3+(a-x)^3}=2a$. 16. $\dfrac{x^4+6x^2+1}{x^4-6x^2+1}=\dfrac{3}{2}$.
17. $\dfrac{x^2-a^2}{x^2+a^2}+\dfrac{x^2+a^2}{x^2-a^2}=\dfrac{34}{15}$.
18. $\dfrac{x^2-5x+3}{x^2+5x-3}-\dfrac{x^2+5x-3}{x^2-5x+3}=\dfrac{8}{3}$.

PROBLEMS.

16. Pr. **1.** The sum of two numbers is 15, and their product is 56. What are the numbers?

Let x stand for one of the numbers; then, by the first condition, $15-x$ stands for the other number. By the second condition

$$x(15-x)=56;\ \text{whence } x=7, \text{ and } 8.$$

Therefore $x=7$, one of the numbers, and $15-x=8$, the other number. Observe that if we take $x=8$, then $15-x=7$. That is, the two required numbers are the two roots of the quadratic equation.

Pr. **2.** Divide 100 into two parts whose product is 2600.

Let x stand for the less part, and $100-x$ for the greater.

By the second condition, $x(100-x)=2600$. The roots of this equation are $50+10\sqrt{-1}$ and $50-10\sqrt{-1}$.

An imaginary result always indicates inconsistent conditions in the problem. The inconsistency of these conditions may be shown as follows:

Let d stand for the difference between the two parts of 100. Then $50 + \frac{1}{2} d$ stands for the greater part, and $50 - \frac{1}{2} d$ for the less.

The product of the two parts is

$$(50 + \tfrac{1}{2} d)(50 - \tfrac{1}{2} d), = 2500 - (\tfrac{1}{2} d)^2 = 2500 - \tfrac{1}{4} d^2.$$

Since d^2 is positive for all *real* values of d, the product $2500 - \frac{1}{4} d^2$ must be less than 2500. Consequently 100 cannot be divided into two parts whose product is greater than 2500.

17. When the solution of a problem leads to a quadratic equation, it is necessary to determine whether either or both of the roots of the equation satisfy the conditions expressed and implied in the problem.

Positive results, in general, satisfy all the conditions of the problem.

A *negative result,* as a rule, satisfies the conditions of the problem, when they refer to abstract numbers. When the required numbers refer to quantities which can be understood in opposite senses, as opposite directions, etc., an intelligible meaning can usually be given to a negative result.

An *imaginary result* always implies inconsistent conditions.

18. The interpretation of a negative result is often facilitated by the following principle:

If a given quadratic equation have a negative root, then the equation obtained by changing the sign of x has a positive root of the same absolute value.

E.g., the roots of the equation $x^2 - 5x + 6 = 0$ are 2 and 3; and the roots of the equation

$$(-x)^2 - 5(-x) + 6 = 0,$$

or $$x^2 + 5x + 6 = 0, \text{ are } -2 \text{ and } -3.$$

Pr. 3. A man bought muslin for \$3.00. If he had bought 3 yards more for the same money, each yard would have cost him 5 cents less. How many yards did he buy?

Let x stand for the number of yards the man bought. Then 1 yard cost $\frac{300}{x}$ cents. If he had bought $x+3$ yards for the same money, each yard would have cost $\frac{300}{x+3}$ cents.

Therefore $\frac{300}{x}-\frac{300}{x+3}=5$; whence $x=12$ and -15.

The root 12 satisfies the equation and also the conditions of the problem; the root -15 has no meaning.

But if x be replaced by $-x$ in the equation, we obtain a new equation,

$$\frac{300}{-x}-\frac{300}{-x+3}=5, \text{ or } \frac{300}{x-3}-\frac{300}{x}=5, \tag{1}$$

whose roots are -12 and $+15$.

Equation (1) evidently corresponds to the problem: A man bought muslin for \$3.00. If he had bought 3 yards less for the same money, each yard would have cost him 5 cents more.

Notice that the intelligible result, 12, of the first statement has become -12 and is meaningless in the second statement.

EXERCISES IX.

1. If 1 be added to the square of a number, the sum will be 50. What is the number?

2. If 5 be subtracted from a number, and 1 be added to the square of the remainder, the sum will be 10. What is the number?

3. One of two numbers exceeds 50 by as much as the other is less than 50, and their product is 2400. What are the numbers?

4. The product of two consecutive integers exceeds the smaller by 17,424. What are the numbers?

5. If 27 be divided by a certain number, and the same number be divided by 3, the results will be equal. What is the number?

6. What number, added to its reciprocal, gives 2.9?

7. What number, subtracted from its reciprocal, gives n? Let $n = 6.09$.

8. If n be divided by a certain number, the result will be the same as if the number were subtracted from n. What is the number? Let $n = 4$.

9. If the product of two numbers be 176, and their difference be 5, what are the numbers?

10. A certain number was to be added to $\frac{1}{2}$, but by mistake $\frac{1}{2}$ was divided by the number. Nevertheless, the correct result was obtained. What was the number?

11. If 100 marbles be so divided among a certain number of boys that each boy shall receive four times as many marbles as there are boys, how many boys are there?

12. The area of a rectangle, one of whose sides is 7 inches longer than the other, is 494 square inches. How long is each side?

13. The difference between the squares of two consecutive numbers is equal to three times the square of the less number. What are the numbers?

14. A merchant received $ 48 for a number of yards of cloth. If the number of dollars a yard be equal to three-sixteenths of the number of yards, how many yards did he sell?

15. In a company of 14 persons, men and women, the men spent $ 24 and the women $ 24. If each man spent $ 1 more than each woman, how many men and how many women were in the company?

16. A pupil was to add a certain number to 4, then to subtract the same number from 9, and finally to multiply the results. But he added the number to 9, then subtracted 4 from the number, and multiplied these results. Nevertheless he obtained the correct product. What was the number?

17. A man paid $ 80 for wine. If he had received 4 gallons less for the same money, he would have paid $ 1 more a gallon. *How many* gallons did he buy?

18. A man left $31,500 to be divided equally among his children. But since 3 of the children died, each remaining child received $3375 more. How many children survived?

19. Two bodies move from the vertex of a right angle along its sides at the rate of 12 feet and 16 feet a second respectively. After how many seconds will they be 90 feet apart?

20. A tank can be filled by two pipes, by the one in two hours less time than by the other. If both pipes be open $1\frac{7}{8}$ hours, the tank will be filled. How long does it take each pipe to fill the tank?

21. From a thread, whose length is equal to the perimeter of a square, 36 inches are cut off, and the remainder is equal in length to the perimeter of another square whose area is four-ninths of that of the first. What is the length of the thread?

22. A number of coins can be arranged in a square, each side containing 51 coins. If the same number of coins be arranged in two squares, the side of one square will contain 21 more coins than the side of the other. How many coins does the side of each of the latter squares contain?

23. A farmer wished to receive $2.88 for a certain number of eggs. But he broke 6 eggs, and in order to receive the desired amount he increased the price of the remaining eggs by $2\frac{2}{5}$ cents a dozen. How many eggs had he originally?

24. Two bodies move toward each other from A and B respectively, and meet after 35 seconds. If it takes the one 24 seconds longer than the other to move from A to B, how long does it take each one to move that distance?

25. It takes a boat's crew 4 hours and 12 minutes to row 12 miles down a river with the current, and back again against the current. If the speed of the current be 3 miles an hour, at what rate can the crew row in still water?

26. A man paid $300 for a drove of sheep. By selling all but 10 of them at a profit of $2.50 each, he received the amount he paid for all the sheep. How many sheep did he buy?

CHAPTER XIX.

SIMULTANEOUS QUADRATIC AND HIGHER EQUATIONS.

1. The solution of a system of quadratic or higher equations in general involves the solution of an equation of higher degree than the second, and therefore cannot be effected by the methods for solving quadratic equations. But there are many special systems whose solutions can be made to depend upon the solutions of quadratic equations.

The following methods are based upon equivalent systems of equations.

2. Elimination by Substitution. — When one equation of a system of two equations is of the first degree, the solution can be obtained by the method of substitution.

Ex. Solve the system
$$\left.\begin{aligned} y + 2x &= 5, \\ x^2 - y^2 &= -8. \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Solving (1) for y, $\qquad y = 5 - 2x. \qquad (3)$

Substituting $5 - 2x$ for y in (2),
$$x^2 - 25 + 20x - 4x^2 = -8. \qquad (4)$$

From this equation we obtain $\qquad x = 1,$

and $\qquad x = 5\frac{2}{3}.$

Substituting 1 for x in (3), $\qquad y = 3.$

Substituting $5\frac{2}{3}$ for x in (3), $\qquad y = -6\frac{1}{3}.$

The equations (3)–(4) are equivalent to the given equations (1)–(2).

Therefore the solutions of the given system are 1, 3; $5\frac{2}{3}$, $-6\frac{1}{3}$, the first number of each pair being the value of x, and the second the corresponding value of y.

Had we substituted 1 for x in (2), we should have obtained $y = \pm 3$.

But the solution 1, -3 does not satisfy equation (1).

Therefore, always substitute in the linear equation the value of the unknown number obtained by elimination.

3. Elimination by Addition and Subtraction. — This method can frequently be applied.

Ex. Solve the system
$$\left.\begin{aligned} x^2 + 3y &= 18, \\ 2x^2 - 5y &= 3. \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

We will first eliminate y.

Multiplying (1) by 5, $\quad 5x^2 + 15y = 90. \qquad (3)$

Multiplying (2) by 3, $\quad 6x^2 - 15y = 9. \qquad (4)$

Adding (3) and (4), $\quad 11x^2 = 99.$

Whence, $\quad x = 3$, and $x = -3$.

Substituting 3 for x in (1), $\quad y = 3.$

Substituting -3 for x in (1), $\quad y = 3.$

The given system has the two solutions 3, 3; -3, 3.

Notice that this example could also have been solved by the method of substitution.

EXERCISES I.

Solve each of the following systems:

1. $\begin{cases} xy = 54, \\ 3x = 2y. \end{cases}$ **2.** $\begin{cases} x^2 + y^2 = 13, \\ x^2 - y^2 = 5. \end{cases}$ **3.** $\begin{cases} x^2 + y^2 = a, \\ x^2 - y^2 = b. \end{cases}$

4. $\begin{cases} 4x - 3y = 24, \\ xy = 96. \end{cases}$ **5.** $\begin{cases} 2x^2 - 3y^2 = 24, \\ 2x = 3y. \end{cases}$ **6.** $\begin{cases} 2x^2 - 3y = 20, \\ x^2 + 5y = 36. \end{cases}$

7. $\begin{cases} 3x - 2y = 1, \\ x^2 + y^2 = 74. \end{cases}$ **8.** $\begin{cases} 7x + xy = 20, \\ 2xy + 5x = 22. \end{cases}$ **9.** $\begin{cases} 2x + 3y = 10, \\ x(x + y) = 25. \end{cases}$

10. $\begin{cases} 4x^2 - xy = 0, \\ 2x - 3y = 6. \end{cases}$ **11.** $\begin{cases} 5xy + 3x^2 = 132, \\ 5xy - 3x^2 = 78. \end{cases}$ **12.** $\begin{cases} 4x = xy + 5, \\ 7y = xy + 6. \end{cases}$

13. $\begin{cases} 3x = x^2 + y^2 - 1, \\ 3y = x^2 + y^2 - 7. \end{cases}$

14. $\begin{cases} x^2 + xy + y^2 = 343, \\ 2x - y = 21. \end{cases}$

15. $\begin{cases} 2x^2 - 3xy + y^2 = 14, \\ 2x - y = 7. \end{cases}$

16. $\begin{cases} x^2 + 5xy + y^2 = 43, \\ x^2 + 5xy - y^2 = 25. \end{cases}$

17. $\begin{cases} 2x - 3y = 11, \\ \frac{4}{x} - \frac{3}{y} = -\frac{17}{7}. \end{cases}$

18. $\begin{cases} x + 2y = 1, \\ \frac{x}{y} + \frac{y}{x} + 3\frac{1}{3} = 0. \end{cases}$

19. $\begin{cases} \frac{x+y}{x-y} + 3x = 2\frac{2}{3}, \\ 5\frac{x+y}{x-y} - 7x = -8\frac{2}{3}. \end{cases}$

20. $\begin{cases} 3x + \sqrt{\frac{x}{y}} = 30, \\ 5x - 2\sqrt{\frac{x}{y}} = 39. \end{cases}$

4. Homogeneous Equations. — When all the terms which contain the unknown numbers in both equations of the system are of the second degree, a system can always be derived whose solution is obtained by the method of Art. 2.

Ex. Solve the system $\quad x^2 + xy + 2y^2 = 74, \quad (1)$

$$2x^2 + 2xy + y^2 = 73. \quad (2)$$

Multiplying (1) by 73, $73x^2 + 73xy + 146y^2 = 74 \times 73. \quad (3)$

Multiplying (2) by 74, $148x^2 + 148xy + 74y^2 = 74 \times 73. \quad (4)$

Subtracting (3) from (4), $75x^2 + 75xy - 72y^2 = 0,$

or $\quad 25x^2 + 25xy - 24y^2 = 0,$

or $\quad (5x - 3y)(5x + 8y) = 0.$

Therefore the given system is equivalent to

$$\left.\begin{aligned} 5x - 3y &= 0, \\ x^2 + xy + 2y^2 &= 74, \end{aligned}\right\}(a), \qquad \left.\begin{aligned} 5x + 8y &= 0, \\ x^2 + xy + 2y^2 &= 74, \end{aligned}\right\}(b).$$

The solutions of these systems, and hence of the given system, are respectively 3, 5; -3, -5; 8, -5; -8, 5.

In applying this method to such systems, we must first derive from the given equations a homogeneous equation in which there is no term free from the unknown numbers.

5. Such examples can also be solved by a special device.

Ex. Solve the system $x^2 + 4y^2 = 13,$ (1)

$xy + 2y^2 = 5.$ (2)

In both equations, let $y = tx.$ (3)

Then from (1), $x^2 + 4x^2t^2 = 13$, whence $x^2 = \dfrac{13}{1+4t^2};$ (4)

and from (2), $x^2t + 2x^2t^2 = 5$, whence $x^2 = \dfrac{5}{t+2t^2}.$ (5)

Equating values of x^2, $\dfrac{13}{1+4t^2} = \dfrac{5}{t+2t^2}.$ (6)

Whence $t = \frac{1}{3}$, and $t = -\frac{5}{2}$.

When $t = \frac{1}{3}$, $x^2 = \dfrac{13}{1+4t^2} = 9$, whence $x = \pm 3$.

When $t = -\frac{5}{2}$, $x^2 = \frac{1}{2}$, whence $x = \pm\sqrt{\frac{1}{2}}$.

When $x = \pm 3$, $y = tx = \frac{1}{3}(\pm 3) = \pm 1$.

When $x = \pm\sqrt{\frac{1}{2}}$, $y = -\frac{5}{2}(\pm\sqrt{\frac{1}{2}}) = \mp\frac{5}{2}\sqrt{\frac{1}{2}}$.

EXERCISES II.

Solve each of the following systems:

1. $\begin{cases} x^2 + xy = 78, \\ y^2 - xy = 7. \end{cases}$

2. $\begin{cases} x^2 + 4y^2 = 13, \\ xy + 2y^2 = 5. \end{cases}$

3. $\begin{cases} x^2 + xy + y^2 = 52, \\ xy - x^2 = 8. \end{cases}$

4. $\begin{cases} x^2 - xy + y^2 = 21, \\ y^2 - 2xy + 15 = 0. \end{cases}$

5. $\begin{cases} x^2 + xy + 4y^2 = 6, \\ 3x^2 + 8y^2 = 14. \end{cases}$

6. $\begin{cases} x^2 - 2xy + 3y^2 = 9, \\ x^2 - 4xy + 5y^2 = 5. \end{cases}$

7. $\begin{cases} x^2 + xy + y^2 = 13x, \\ x^2 - xy + y^2 = 7x. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 61 - 3xy, \\ x^2 - y^2 = 31 - 2xy. \end{cases}$

6. Symmetrical Equations. — A **Symmetrical Equation** is one which remains the same when the unknown numbers are interchanged.

A system of two symmetrical equations can be solved by first finding the values of $x + y$ and $x - y$.

Ex. 1. Solve the system
$$\left.\begin{aligned} x^2+y^2&=13,\\ xy&=6.\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiplying (2) by 2, $\quad 2xy=12.\qquad (3)$

Adding (3) to (1), $\quad x^2+2xy+y^2=25.\qquad (4)$

Subtracting (3) from (1), $\quad x^2-2xy+y^2=1.\qquad (5)$

Equating square roots of (4), $\quad x+y=\pm 5.\qquad (6)$

Equating square roots of (5), $\quad x-y=\pm 1.\qquad (7)$

Equations (4)–(5), or (6)–(7), are equivalent to (1)–(2). But (6) and (7) are equivalent to

$$\left.\begin{aligned}x+y&=5,\\ x-y&=1,\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=5,\\ x-y&=-1,\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=-5,\\ x-y&=+1,\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=-5,\\ x-y&=-1.\end{aligned}\right\}.$$

The solutions of these four systems are respectively 3, 2; 2, 3; -2, -3; -3, -2.

The solutions of (6) and (7) should be obtained mentally, without writing the equivalent systems. Each sign of the second member of (6) should be taken in turn with each sign of the second member of (7).

Notice that these solutions differ only in having the values or x and y interchanged. This we should expect from the definition of symmetrical equations.

When the equations are symmetrical, except for sign, the solution can be obtained by a similar method.

Ex. 2. Solve the system

$$x-y=3,\qquad (1)$$

$$x^2+y^2=29.\qquad (2)$$

Squaring (1), $\quad x^2-2xy+y^2=9,\qquad (3)$

Subtracting (3) from (2),

$$2xy=20,\text{ or } xy=10.\qquad (4)$$

The solutions of (1) and (4) are 5, 2; -2, -5.

Notice that the solutions in this case differ not only in *having* the values of x and y interchanged, but also in sign.

EXERCISES III.

Solve each of the following systems:

1. $\begin{cases} x+y=12, \\ xy=32. \end{cases}$ **2.** $\begin{cases} x+y=a, \\ xy=b. \end{cases}$ **3.** $\begin{cases} \frac{1}{2}x+5y=37, \\ xy=28. \end{cases}$

4. $\begin{cases} x-y=8, \\ xy=-15. \end{cases}$ **5.** $\begin{cases} x-y=m, \\ xy=n. \end{cases}$ **6.** $\begin{cases} 6x-7y=58, \\ 3xy=-60. \end{cases}$

7. $\begin{cases} x^2+y^2=40, \\ xy=12. \end{cases}$ **8.** $\begin{cases} x^2+y^2=181, \\ xy=-90. \end{cases}$ **9.** $\begin{cases} 25x^2+9y^2=148, \\ 5xy=8. \end{cases}$

10. $\begin{cases} 9x^2+y^2=37a^2, \\ xy=-2a^2. \end{cases}$ **11.** $\begin{cases} 5x^2+2y^2=5a^2+8b^2, \\ xy=2ab. \end{cases}$

12. $\begin{cases} x^2+y^2=137, \\ x+y=15. \end{cases}$ **13.** $\begin{cases} x^2+y^2=61, \\ x+y=11. \end{cases}$ **14.** $\begin{cases} 5x+3y=11, \\ 25x^2+9y^2=73. \end{cases}$

15. $\begin{cases} x^2-y^2=28, \\ xy=48. \end{cases}$ **16.** $\begin{cases} x^2-4y^2=-3, \\ xy=-1. \end{cases}$ **17.** $\begin{cases} x^2+y^2=53, \\ x-y=5. \end{cases}$

18. $\begin{cases} x^2+y^2=74, \\ x-y=2. \end{cases}$ **19.** $\begin{cases} 9x^2+y^2=82, \\ 3x-y=10. \end{cases}$ **20.** $\begin{cases} 16x^2+49y^2=113, \\ 4x+7y=1. \end{cases}$

21. $\begin{cases} xy=80, \\ \frac{1}{x}-\frac{1}{y}=\frac{1}{5}. \end{cases}$ **22.** $\begin{cases} x+y=16, \\ \frac{1}{x}+\frac{1}{y}=\frac{1}{3}. \end{cases}$ **23.** $\begin{cases} x^2+y^2=2\frac{1}{2}xy, \\ \frac{1}{x}+\frac{1}{y}=1\frac{1}{2}. \end{cases}$

24. $\begin{cases} \frac{1}{x}+\frac{1}{y}=3, \\ \frac{1}{xy}=2. \end{cases}$ **25.** $\begin{cases} \frac{x}{y}-\frac{y}{x}=\frac{16}{15}, \\ x-y=2. \end{cases}$ **26.** $\begin{cases} \frac{1}{x}+\frac{1}{y}=10, \\ \frac{1}{x^2}+\frac{1}{y^2}=58. \end{cases}$

27. $\begin{cases} x+xy+y=29, \\ x^2+xy+y^2=61. \end{cases}$ **28.** $\begin{cases} x^2+y^2+7xy=171, \\ xy=2(x+y). \end{cases}$

29. $\begin{cases} x^2+y^2-(x-y)=20, \\ xy+x-y=1. \end{cases}$ **30.** $\begin{cases} x^2+y^2-x-y=22, \\ x+y+xy=-1. \end{cases}$

31. $\begin{cases} x^2+y^2+x-y=a, \\ xy+x-y=b. \end{cases}$ **32.** $\begin{cases} x+y=2, \\ x^2+y^2+xy=3. \end{cases}$

33. $\begin{cases} x+y=9, \\ x^2+y^2-xy=21. \end{cases}$ **34.** $\begin{cases} x^2+xy+y^2=2m, \\ x^2-xy+y^2=2n. \end{cases}$

7. Higher Equations. — The solutions of certain equations of higher degree than the second can be made to depend upon the solutions of quadratic equations.

Ex. 1. Solve the system $x^3 + y^3 = 9,$ (1)

$$x + y = 3. \quad (2)$$

Dividing (1) by (2), $x^2 - xy + y^2 = 3.$ (3)

Subtracting (3) from the square of (2),

$$3xy = 6, \text{ or } xy = 2. \quad (4)$$

The solutions of (2) and (4), and therefore of the given system, are 1, 2, and 2, 1.

Ex. 2. Solve the system $x^4 + y^4 = 17,$ (1)

$$x + y = 3. \quad (2)$$

We first find the value of xy.

Let $$xy = z. \quad (3)$$

Squaring (2), $$x^2 + 2xy + y^2 = 9, \quad (4)$$

or $$x^2 + y^2 = 9 - 2z. \quad (5)$$

Squaring (5), $$x^4 + 2x^2y^2 + y^4 = 81 - 36z + 4z^2, \quad (6)$$

or $$x^4 + y^4 = 81 - 36z + 2z^2. \quad (7)$$

Since $x^4 + y^4 = 17$, we have from (7),

$$2z^2 - 36z + 81 = 17. \quad (8)$$

Whence $$z = 16, \text{ and } 2. \quad (9)$$

Therefore, from (3) and (9), $$xy = 16, \quad (10)$$

and $$xy = 2. \quad (11)$$

The solutions of (2) and (10) and of (2) and (11) are readily found, and should be checked by substitution.

EXERCISES IV.

Solve each of the following systems:

1. $\begin{cases} x + y = 5, \\ x^3 + y^3 = 35. \end{cases}$

2. $\begin{cases} x - y = 1, \\ x^3 - y^3 = 7. \end{cases}$

3. $\begin{cases} 2(x + y) = 5, \\ 32(x^3 + y^3) = 2285. \end{cases}$

4. $\begin{cases} (x - 1)^3 + (y - 2)^3 = 28, \\ x + y = 7. \end{cases}$

5. $\begin{cases}(x-7)^3+(5-y)^3=9,\\ x-y=5.\end{cases}$

6. $\begin{cases}x^4-y^4=544,\\ x^2+y^2=34.\end{cases}$

7. $\begin{cases}x^4+y^4=82,\\ xy=3.\end{cases}$

8. $\begin{cases}x^4+y^4=97,\\ x+y=5.\end{cases}$

9. $\begin{cases}x^4+y^4=257,\\ x-y=3.\end{cases}$

10. $\begin{cases}(x-7)^4+(y-3)^4=257,\\ x-y+1=0.\end{cases}$

11. $\begin{cases}(x^2-y^2)(x+y)=9,\\ xy(x+y)=6.\end{cases}$

12. $\begin{cases}(x+y)(x^2+y^2)=175,\\ (x-y)(x^2-y^2)=7.\end{cases}$

13. $\begin{cases}x^4+y^4=14\,x^2y^2,\\ x+y=a.\end{cases}$

14. $\begin{cases}x^3y^2-x^2y^3=1152,\\ x^2y-xy^2=48.\end{cases}$

Problems.

8. Pr. The front wheel of a carriage makes 6 more revolutions than the hind wheel in travelling 360 feet. But if the circumference of each wheel were 3 feet greater, the front wheel would make only 4 revolutions more than the hind wheel in travelling the same distance as before. What are the circumferences of the two wheels?

Let x stand for the number of feet in the circumference of front wheel, and y for the number of feet in the circumference of hind wheel. Then in travelling 360 feet the front wheel makes $\frac{360}{x}$ revolutions, and the hind wheel makes $\frac{360}{y}$ revolutions.

By the first condition, $\frac{360}{x}=\frac{360}{y}+6.$ (1)

If 3 feet were added to the circumference of each wheel, the front wheel would make $\frac{360}{x+3}$ revolutions, and the hind wheel $\frac{360}{y+3}$ revolutions.

By the second condition, $\frac{360}{x+3}=\frac{360}{y+3}+4.$ (2)

Whence $x=12$, the circumference of the front wheel, and $y=15$, the circumference of the hind wheel.

EXERCISES V.

1. The square of one number increased by ten times a second number is 84, and is equal to the square of the second number increased by ten times the first.

2. The sum of two numbers is 20, and the sum of the square of the one diminished by 13 and the square of the other increased by 13 is 272. What are the numbers?

3. Find two numbers such that their difference added to the difference of their squares shall be 150, and their sum added to the sum of their squares shall be 330.

4. Find two numbers whose sum is equal to their product and also to the difference of their squares.

5. The sum of the fourth powers of two numbers is 1921, and the sum of their squares is 61. What are the numbers?

6. If a number of two digits be multiplied by its tens' digit, the product will be 390. If the digits be interchanged and the resulting number be multiplied by its tens' digit, the product will be 280. What is the number?

7. If a number of two digits be divided by the product of its digits, the quotient will be 2. If 27 be added to the number, the sum will be equal to the number obtained by interchanging the digits. What is the number?

8. The product of the two digits of a number is equal to one-half of the number. If the number be subtracted from the number obtained by interchanging the digits, the remainder will be equal to three-halves of the product of the digits of the number. What is the number?

9. If the difference of the squares of two numbers be divided by the first number, the quotient and the remainder will each be 5. If the difference of the squares be divided by the second number, the quotient will be 13 and the remainder 1. What are the numbers?

10. The sum of the three digits of a number is 9. If the digits be written in reverse order, the resulting number will exceed the original number by 396. The square of the middle digit exceeds the product of the first and third digit by 4. What is the number?

11. A rectangular field is 119 yards long and 19 yards wide. How many yards must be added to its width and how many yards must be taken from its length, in order that its area may remain the same while its perimeter is increased by 24 yards?

12. The floor of a room contains $30\frac{1}{8}$ square yards; one wall contains 21 square yards, and an adjacent wall contains 13 square yards. What are the dimensions of the room?

13. A merchant bought a number of pieces of cloth of two different kinds. He bought of each kind as many pieces and paid for each yard half as many dollars as that kind contained yards. He bought altogether 19 pieces and paid for them $921.50. How many pieces of each kind did he buy?

14. The diagonal of a rectangle is $20\frac{2}{5}$ feet. If the length of one side be increased by 14 feet and the length of the other side be diminished by $2\frac{2}{5}$ feet, the diagonal will be increased by $12\frac{2}{5}$ feet. What are the lengths of the sides of the rectangle?

15. A certain number of coins can be arranged in the form of one square, and also in the form of two squares. In the first arrangement each side of the square contains 29 coins, and in the second arrangement one square contains 41 more coins than the other. How many coins are there in a side of each square of the second arrangement?

16. A piece of cloth after being wet shrinks in length by one-eighth and in breadth by one-sixteenth. The piece contains after shrinking 3.68 fewer square yards than before shrinking, and the length and breadth together shrink 1.7 yards. What was the length and breadth of the piece?

17. A merchant paid $125 for two kinds of goods. He sold the one kind for $91 and the other for $36. He thereby

gained as much per cent on the first kind as he lost on the second. How much did he pay for each kind?

18. Two workmen can do a piece of work in 6 days. How long will it take each of them to do the work, if it takes one 5 days longer than the other?

19. Two men, A and B, receive different wages. A earns \$42, and B \$40. If A had received B's wages a day, and B had received A's wages, they would have earned together \$4 more. How many days does each work, if A works 8 days more than B, and what wages does each receive?

20. It takes a number of workmen 8 hours to remove a pile of stones from one place to another. Had there been 8 more workmen, and had each one carried 5 pounds less at each trip, they would have completed the work in 7 hours. Had there been 8 fewer workmen and had each one carried 11 pounds more at each trip, they would have completed the work in 9 hours. How many workmen were there and how many pounds did each one carry at every trip?

21. A tank can be filled by one pipe and emptied by another. If, when the tank is half full of water, both pipes be left open 12 hours, the tank will be emptied. If the pipes be made smaller, so that it will take the one pipe one hour longer to fill the tank and the other one hour longer to empty it, the tank, when half full of water, will then be emptied in $15\frac{3}{4}$ hours. In what time will the empty tank be filled by the one pipe, and the full tank be emptied by the other?

CHAPTER XX.

RATIO, PROPORTION, AND VARIATION.

RATIO.

1. The **Ratio** of one number to another is the relation between the numbers which is expressed by the quotient of the first divided by the second.

E.g., the ratio of 6 to 4 is expressed by $\frac{6}{4}, = \frac{3}{2}$.

The ratio of one number to another is frequently expressed by placing a colon between them; as $5:7$.

The first number in a ratio is called the **First Term**, or the **Antecedent** of the ratio, and the second number the **Second Term**, or the **Consequent** of the ratio.

Thus, in the ratio $a:b$, a is the first term, and b the second.

2. Since, by definition, a ratio is a fraction, all the properties of fractions are true of ratios; as $a:b = ma:mb$.

3. The definition given in Art. 1 has reference to the ratio of one *number* to another. But it is frequently necessary to compare concrete quantities, as the length of one line with the length of another line, etc.

If two concrete quantities of the same kind can be expressed by two rational numbers in terms of the same unit, then the ratio of the one quantity to the other is defined as the ratio of the one number to the other.

E.g., the ratio of $2\frac{1}{2}$ yards to $1\frac{1}{7}$ yards is $2\frac{1}{2}:1\frac{1}{7}, = \frac{2\frac{1}{2}}{1\frac{1}{7}} = \frac{35}{16}$.

Observe that by this definition the ratio of two concrete quantities is a number. Also that the quantities to be compared must be of the same kind. Dollars cannot be compared with pounds, etc.

4. If two concrete quantities cannot be expressed by two rational numbers, integers or fractions, in terms of the same unit, they are said to be **Incommensurable** one to the other.

Thus, if the lengths of the two sides of a right triangle be equal, the length of the hypothenuse cannot be expressed by a rational number in terms of a side as a unit, or any fraction of a side as a unit.

If a side be taken as the unit, the hypothenuse is expressed by $\sqrt{2}$, an irrational number. And the ratio of the hypothenuse to a side is $\sqrt{2}:1, =\sqrt{2}$. But as was shown in Ch. XV, Art. 40, an approximate value of $\sqrt{2}$ can be found to any required degree of accuracy.

5. In general let P and Q be two incommensurable quantities. Then two rational numbers $\frac{m}{n}$ and $\frac{m+1}{n}$ can be found, between which the value of the ratio $P:Q$ lies. These two fractions differ by $\frac{1}{n}$. Therefore, the ratio $P:Q$, which lies between them, differs from either of them by less than $\frac{1}{n}$. By taking n sufficiently great we can make $\frac{1}{n}$ as small as we please, that is, *less than any assigned number, however small.*

It can be proved that the ratio of two incommensurable quantities is a number which obeys the fundamental laws of algebra.

It is therefore not necessary, in the principles of this chapter, to make any distinction between such ratios and those which can be expressed exactly in terms of integers and fractions.

EXERCISES I.

What is the ratio of

1. $6a$ to $9b$?
2. $\frac{3}{5}a^2b$ to $\frac{6}{11}ab^2$?
3. $9\frac{1}{2}x^3y$ to $7\frac{3}{5}xy^3$?
4. $\frac{1}{a}$ to $\frac{1}{b}$?
5. $\frac{a}{b}$ to $\frac{c}{d}$?
6. $\frac{a}{x-3}$ to $\frac{1}{(x-3)^2}$?

7. Which is the greater ratio,

$$a + 2b : a + b \text{ or } a + 3b : a + 2b?$$

What is the value of the ratio $x : y$

8. If $\dfrac{6x + 2y}{3x - y} = 10$? **9.** If $\dfrac{8x + 4y}{3x - 2y} = 5$?

If the value of the ratio $x : y$ is $\frac{3}{5}$, what is the value

10. Of $\dfrac{10x - y}{15x + y}$? **11.** Of $\dfrac{5x + 6y}{3x - 2y}$?

PROPORTION.

6. A **Proportion** is an equation whose members are two equal ratios.

E.g., $4 : 3 = 8 : 6$, read *the ratio of* 4 *to* 3 *is equal to the ratio of* 8 *to* 6, or 4 *is to* 3 *as* 8 *is to* 6.

Instead of the equality sign a double colon is frequently used; as $4 : 3 :: 8 : 6$.

7. Four numbers are said to be *in proportion*, or to be *proportional*, when the first is to the second as the third is to the fourth.

E.g., the numbers 4, 3, 8, 6 are proportional, since $4 : 3 = 8 : 6$.

The individual numbers are called the **Proportionals**, or **Terms** of the proportion.

The **Extremes** of a proportion are its first and last terms; as 4 and 6 above.

The **Means** of a proportion are its second and third terms; as 3 and 8 above.

The **Antecedents** and **Consequents** of a proportion are the antecedents and consequents of its two ratios.

E.g., 4 and 8 are the antecedents, and 3 and 6 the consequents of the proportion $4 : 3 = 8 : 6$.

Principles of Proportions.

8. *In any proportion the product of the extremes is equal to the product of the means.*

If $a : b = c : d$, we are to prove $ad = bc$.

By Art. 1, $\frac{a}{b} = \frac{c}{d}$.

Clearing of fractions, $ad = bc$.

9. *If the product of two numbers be equal to the product of two other numbers, the four numbers are in proportion.*

Let $ad = bc$.

Dividing by bd, $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$; (1)

by cd, $\frac{a}{c} = \frac{b}{d}$, or $a : c = b : d$; (2)

by ab, $\frac{d}{b} = \frac{c}{a}$, or $d : b = c : a$; (3)

by ac, $\frac{d}{c} = \frac{b}{a}$, or $d : c = b : a$. (4)

Interchanging the ratios in (1), (2), (3), (4),

$c : d = a : b$; (5)

$b : d = a : c$; (6)

$c : a = d : b$; (7)

$b : a = d : c$. (8)

Notice that the two numbers of either product may be taken as the extremes, the other two as the means. In (1) to (4), a and d are the extremes, c and b the means; in (5) to (8), d and a are the means, c and b the extremes.

10. In Art. 9, we may regard the proportions (2) to (8) as being derived from (1), and thus obtain the following properties of a proportion:

(i.) *The means may be interchanged;* as in (2).

(ii.) *The extremes may be interchanged;* as in (3).

(iii.) *The means may be interchanged, and at the same time the extremes;* as in (4).

(iv.) *The means may be taken as the extremes, and the extremes as the means;* as (8) from (1), (7) from (2), etc.

11. *If any three terms of a proportion be given, the remaining term can be found.*

Ex. What is the second term of a proportion, whose first, third, and fourth terms are 10, 16, and 8 respectively?

Letting x stand for the second term, we have

$$10 : x = 16 : 8, \text{ or } 16x = 80; \text{ whence } x = 5.$$

12. *The products, or the quotients, of the corresponding terms of two proportions form again a proportion.*

If $$a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d}, \tag{1}$$

and $$x : y = z : u, \text{ or } \frac{x}{y} = \frac{z}{u}, \tag{2}$$

we have, multiplying corresponding members of (1) and (2),

$$\frac{ax}{by} = \frac{cz}{du}; \text{ whence } ax : by = cz : du.$$

Dividing the members of (1) by the corresponding members of (2), we have

$$\frac{\frac{a}{x}}{\frac{b}{y}} = \frac{\frac{c}{z}}{\frac{d}{u}}; \text{ whence } \frac{a}{x} : \frac{b}{y} = \frac{c}{z} : \frac{d}{u}.$$

13. *In any proportion, the sum of the first two terms is to the first (or the second) term as the sum of the last two terms is to the third (or the fourth) term.*

Let $$a : b = c : d.$$

Then $$\frac{a}{b} = \frac{c}{d}.$$

Adding 1 to both members, $\frac{a}{b} + 1 = \frac{c}{d} + 1,$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

Whence $$a+b:b=c+d:d.$$

In like manner it can be proved that

$$a+b:a=c+d:c.$$

These two proportions are said to be derived from the given proportion by **Composition.**

14. *In any proportion, the difference of the first two terms is to the first (or the second) term as the difference of the last two terms is to the third (or the fourth) term.*

If $$a:b=c:d,$$

then $$a-b:a=c-d:c, \text{ and } a-b:b=c-d:d.$$

The proof is similar to that of Art. 13.

These two proportions are said to be derived from the given proportion by **Division.**

15. *In any proportion, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Let $$a:b=c:d.$$

By Art. 13, $$a+b:b=c+d:d;$$

and by Art. 14, $$a-b:b=c-d:d.$$

Then by Art. 12, $$\frac{a+b}{a-b}:1=\frac{c+d}{c-d}:1,$$

or $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Whence $$a+b:a-b=c+d:c-d.$$

This proportion is said to be derived from the given one by **Composition and Division.**

16. A **Continued Proportion** is one in which the consequent of each ratio is the antecedent of the following ratio; as,

$$a:b=b:c=c:d=\text{etc.}$$

17. In the continued proportion

$$a : b = b : c,$$

b is called a **Mean Proportional** between a and c, and c is called the **Third Proportional** to a and b.

18. *The mean proportional between any two numbers is equal to the square root of their product.*

From $$a : b = b : c,$$

we have, by Art. 8, $b^2 = ac$; whence $b = \sqrt{(ac)}$.

19. *In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $$n_1 : d_1 = n_2 : d_2 = n_3 \; d_3 = \cdots = v,$$

or $$\frac{n_1}{d_1} = v, \; \frac{n_2}{d_2} = v, \; \frac{n_3}{d_3} = v, \; \cdots.$$

Then $$n_1 = vd_1, \; n_2 = vd_2, \; n_3 = vd_3, \; \cdots.$$

Ad ing corresponding members of these equations, we have

$$n_1 + n_2 + n_3 + \cdots = vd_1 + vd_2 + vd_3 + \cdots$$
$$= v(d_1 + d_2 + d_3 + \cdots).$$

Therefore $$\frac{n_1 + n_2 + n_3 + \cdots}{d_1 + d_2 + d_3 + \cdots} = v = \frac{n_1}{d_1} = \frac{n_2}{d_2} = \cdots.$$

E.g., $$\frac{1}{2} = \frac{4}{8} = \frac{5}{10} = \frac{1 + 4 + 5}{2 + 8 + 10} = \frac{10}{20}.$$

20. The following examples are applications of the preceding theory:

Ex. 1. Find a mean proportional between 5 and 20.

Let x stand for the required proportional.

Then, by Art. 18, $x = \sqrt{(5 \times 20)} = \pm 10$.

Ex. 2. If $$a : b = c : d,$$

then $$ab + cd : ab - cd = b^2 + d^2 : b^2 - d^2.$$

Let $$\frac{a}{b} = \frac{c}{d} = x.$$

Then $a = bx$ and $c = dx$.

Therefore $ab + cd = b^2x + d^2x$,

and $ab - cd = b^2x - d^2x$.

We then have $\dfrac{ab+cd}{ab-cd} = \dfrac{b^2x + d^2x}{b^2x - d^2x} = \dfrac{b^2 + d^2}{b^2 - d^2}$.

Whence $ab + cd : ab - cd = b^2 + d^2 : b^2 - d^2$.

Ex. 3. Solve the equation

$$\frac{\sqrt{(2+x)} + \sqrt{(2-x)}}{\sqrt{(2+x)} - \sqrt{(2-x)}} = 2, = \frac{2}{1}.$$

By composition and division,

$$\frac{\sqrt{(2+x)}}{\sqrt{(2-x)}} = \frac{3}{1}.$$

Squaring and clearing of fractions,

$$2 + x = 18 - 9x; \text{ whence } x = \tfrac{8}{5}.$$

EXERCISES II.

Verify each of the following proportions:

1. $2\frac{1}{2} : 1\frac{1}{3} = 1\frac{1}{2} : \frac{4}{5}$.

2. $14\frac{2}{3} : 4\frac{2}{5} = 200 : 60$.

3. $\dfrac{4ab}{a^2 - b^2} : \dfrac{a^2 + b^2}{a - b} = \dfrac{2ab}{a^4 - b^4} : \dfrac{1}{2a - 2b}$.

Form proportions from each of the following products, in eight different ways:

4. $2x = 3y$.

5. $m^2 = n^2$.

6. $a^3 - b^3 = x^2 - y^2$.

Find a fourth proportional to

7. 1, 2, and 8.

8. $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$.

9. ab, ac, and b.

Find a third proportional to

10. 2 and 6.

11. $\frac{1}{3}$ and $\frac{1}{6}$.

12. a and b.

Find a mean proportional between

13. 2 and 18.

14. $\frac{1}{3}$ and $\frac{3}{4}$.

15. a^2b and ab^2.

16. $\dfrac{a+b}{a-b}$ and $\dfrac{a^2-b^2}{a^2b^2}$.

17. $\dfrac{a^2+1}{a^2-1}$ and $\frac{1}{4}(a^4 - 1)$.

Find the value of x to satisfy each of the following proportions:

18. $x : 2 = 12 : 3.$ **19.** $161 : 253 = x : 407.$ **20.** $7\frac{1}{3} : 1\frac{4}{7} = \frac{7}{9} : x.$

21. $\frac{1}{2} + \sqrt{a} : \frac{1}{4} - a = x : \sqrt{a} - 2a.$

22. $a + b + \dfrac{2b^2}{a-b} : \dfrac{(a+b)^2}{2ab} - 1 = x : a - b.$

Solve each of the following equations:

23. $\dfrac{\sqrt{(1+x)} + \sqrt{(1-x)}}{\sqrt{(1+x)} - \sqrt{(1-x)}} = 3.$ **24.** $\dfrac{\sqrt{(a+x)} - \sqrt{(a-x)}}{\sqrt{(a+x)} + \sqrt{(a-x)}} = \dfrac{1}{\sqrt{b}}.$

25. $\dfrac{\sqrt{(ax)} + b}{\sqrt{(ax)} - b} = \dfrac{a+b}{a-b}.$ **26.** $\dfrac{\sqrt{a} + \sqrt{(bx)}}{a+b} = \dfrac{\sqrt{a} - \sqrt{(bx)}}{a-b}.$

27. $\dfrac{5x+6}{5x-7} = \dfrac{7x+4}{7x-9}.$ **28.** $\dfrac{x^2 - x + 6}{x^2 + x - 6} = \dfrac{x^2 + 2x - 3}{x^2 - 2x + 3}.$

29. $\dfrac{x^2 - 5x + 4}{4x - 4} = \dfrac{x^2 - 3x + 2}{3x - 2}.$ **30.** $\dfrac{x^2 - 4x + 3}{4x - 3} = \dfrac{x^2 - 6x + 7}{6x - 7}.$

31. Find two numbers whose ratio is $7 : 5$, and the difference of whose squares is 96.

32. A works 6 days with 2 horses, and B works 5 days with 3 horses. What is the ratio of A's work to B's work?

33. The ratio of a father's age to his son's age is $9 : 5$. If the father is 28 years older than the son, how old is each?

34. Find three numbers in a continued proportion whose sum is 39, and whose product is 729.

35. Find two numbers such that if one be added to the first and 8 to the second, the sums will be in the ratio $1 : 2$, and if 1 be subtracted from each number, the remainders will be in the ratio $2 : 3$.

36. What is the ratio of the numerator of a fraction to its denominator, if the fraction be unchanged when a is added to its numerator and b to its denominator?

37. The sum of the means of a proportion is 7, the sum of the extremes is 8, and the sum of the squares of all the terms is 65. What is the proportion?

If $a : b = c : d$, prove that

38. $a + c : b + d = a^2d : b^2c$.

39. $a^2 + b^2 : a^2 - b^2 = c^2 + d^2 : c^2 - d^2$.

40. $(a \pm b)^2 : ab = (c \pm d)^2 : cd$.

41. $2a + 3b : 4a + 5b = 2c + 3d : 4c + 5d$.

42. $a + b : c + d = \sqrt{(a^2 + b^2)} : \sqrt{(c^2 + d^2)}$.

43. $\sqrt{(a^2 + b^2)} : \sqrt{(c^2 + d^2)} = \sqrt[3]{(a^3 + b^3)} : \sqrt[3]{(c^3 + d^3)} = a : c$.

VARIATION.

21. Frequently two numbers or quantities are so related to each other that a change in the value of one produces a corresponding change in the value of the other.

Thus, the distance a train runs in one hour depends upon its speed, and increases or decreases when its speed increases or decreases.

The illumination made by a light depends upon the intensity of the light, and varies when the intensity varies.

The value of y given by the equation $y = 2x - 3$ depends upon the value of x, and varies when the value of x varies.

Thus, if $x = 1$, $y = -1$; if $x = 2$, $y = 1$, etc.

We shall in this chapter consider only the simplest kinds of variation.

22. Direct Variation. — Two quantities are said to *vary directly* one as the other, when their ratio is constant.

Thus, if x varies directly as y, then $\frac{x}{y} = k$, a constant.

For example, if a train runs at a uniform speed, the number of miles it runs varies directly as the number of hours. If it runs at the rate of 30 miles an hour, in 1 hour it will run 30 miles, in 2 hours 60 miles, in 3 hours 90 miles, and so on; and the ratios $1:30$, $2:60$, $3:90$, etc., are equal.

The symbol of direct variation, $\propto$, is read *varies directly as*.

The word *directly* is frequently omitted.

If $y = 3x$, then $y \propto x$ (read *y varies as x*), since $\frac{y}{x} = 3$, a constant.

23. Inverse Variation. — One quantity is said to *vary inversely* as another when the first varies as the *reciprocal* of the second.

Thus, if x varies inversely as y, then $x \propto \frac{1}{y}$.

Therefore, $\frac{x}{\frac{1}{y}} = k$, a constant; whence $xy = k$.

That is, if one quantity varies inversely as another, the product of the quantities is constant.

If 6 men can do a piece of work in 12 hours, 3 men can do the same work in 24 hours, and 1 man in 72 hours, and the products 6×12, 3×24, 1×72 are equal. That is, the number of hours varies inversely as the number of men working.

If $y = \frac{3}{x}$, y varies inversely as x, since $xy = 3$.

24. Joint Variation. — One quantity is said to *vary* as two others *jointly*, when it varies as the product of the others.

Thus, if x varies as y and z jointly, then $\frac{x}{yz} = k$, a constant.

For example, the number of miles a train runs varies as the number of hours and the number of miles it runs an hour jointly. It will run 40 miles in 2 hours at a rate of 20 miles an hour, 90 miles in 3 hours at the rate of 30 miles an hour,

and $$\frac{40}{2 \times 20} = \frac{90}{3 \times 30} = \frac{120}{5 \times 24}.$$

25. One quantity is said to vary directly as a second and inversely as a third, when it varies as the second and the reciprocal of the third jointly.

Thus, if x varies directly as y and inversely as z, then

$\frac{x}{y \cdot \frac{1}{z}} = k$, a constant; or $\frac{xz}{y} = k$.

26. In all the preceding cases of variation, the constant can be determined when any set of corresponding values of the quantities is known.

Ex. 1. If $x \propto y$, and $x = 3$ when $y = 5$, what is the value of the constant?

We have $\frac{x}{y} = k$, or $x = ky$.

Therefore, when $x = 3$ and $y = 5$,

$3 = 5k$, whence $k = \frac{3}{5}$.

Consequently $x = \frac{3}{5}y$.

EXERCISES III.

1. If $x \propto y$, and $x = 10$ when $y = 5$, what is the value of x when $y = 12\frac{1}{2}$?

2. If $x \propto y$, and $x = a$ when $y = \frac{3}{4}a^2$, what is the value of y when $x = a^2b$?

3. If $x \propto y^2$, and $x = 5$ when $y = -3$, what is the value of x when $y = 15$?

4. If $x \propto \sqrt{y}$, and $x = a + m$ when $y = (a - m)^2$, what is the value of x when $y = (a + m)^4$?

5. If $x \propto \frac{1}{y}$, and $x = 3$ when $y = \frac{7}{8}$, what is the value of x when $y = 4\frac{1}{7}$?

6. If $x \propto \frac{y}{z}$, and $x = 4$ when $y = 6$ and $z = 3$, what is the value of x when $y = 5$, and $z = 2$?

7. The circumference of a circle whose radius is 6 feet is 37.7 feet. What is the circumference of a circle whose radius is 9.5 feet, if it be known that the circumference varies as the radius?

8. An ox is tied by a rope 20 yards long in the centre of a field, and eats all the grass within his reach in $2\frac{1}{2}$ days. How many days would it have taken the ox to eat all the grass within his reach if the rope had been 10 yards longer?

9. The volume of a sphere whose radius is 7 inches is 1437.3 cubic inches. What is the volume of a sphere whose radius is 10 inches, if it be known that the volume varies as the cube of the radius?

It has been found by experiment that the distance a body falls from rest varies as the square of the time.

10. If a body falls 256 feet in 4 seconds, how far will it fall in 10 seconds?

11. From what height must a body fall to reach the earth after 15 seconds?

It has been found by experiment that the velocity acquired by a body falling from rest varies as the time.

12. If the velocity of a falling body is 160 feet after 5 seconds, what will be the velocity after 8 seconds?

13. How long must a body have been falling to have acquired a velocity of 384 feet?

14. The surface of a cube whose edge is 5 inches is 150 square inches. What is the surface of a cube whose edge is 9 inches, if it be known that the surface varies as the square of its edge?

15. It has been found by experiment that the weight of a body varies inversely as the square of its distance from the centre of the earth. If a body weighs 30 pounds on the surface of the earth (approximately 4000 miles from the centre), what would be its weight at a distance of 24,000 miles from the surface of the earth?

It has been found by experiment that the illumination of an object varies inversely as the square of its distance from the source of light.

16. If the illumination of an object at a distance of 10 feet from a source of light is 2, what is the illumination at a distance of 40 feet?

17. To what distance must an object which is now 10 feet from a source of light be removed in order that it shall receive only one-half as much light?

18. At what distance will a light of intensity 10 give the same illumination as a light of intensity 8 gives at a distance of 50 feet?

CHAPTER XXI.

PROGRESSIONS.

1. A **Series** is a succession of numbers, each formed according to some definite law. The single numbers are called the **Terms** of the series.

E.g., in the series

$$1 + 3 + 5 + 7 + 9 + \cdots \qquad (1)$$

each term after the first is formed by adding 2 to the preceding term.

In the series $$1 + 2 + 4 + 8 + \cdots \qquad (2)$$

each term after the first is formed by multiplying the preceding term by 2.

2. The number of terms in a series may be either *limited* or *unlimited.*

A **Finite** series is one of a *limited* number of terms.

An **Infinite** series is one of an *unlimited* number of terms.

In this chapter a few simple and yet very important series will be discussed.

ARITHMETICAL PROGRESSION.

3. An **Arithmetical Series**, or, as it is more commonly called, an **Arithmetical Progression** (A. P.), is a series in which each term, after the first, is formed by adding a constant number to the preceding term. See Art. 1, (1).

4. Evidently this definition is equivalent to the statement, that the difference between any two consecutive terms is constant.

E.g., in the series

$$1 + 3 + 5 + 7 + \cdots$$

we have $$3 - 1 = 5 - 3 = 7 - 5 = \cdots$$

For this reason the constant number of the first definition is called the **Common Difference** of the series.

5. Let a_1 stand for the first term of the series,
a_n for the nth (*any*) term of the series,
d for the common difference,
and S_n for the sum of n terms of the series.

The five numbers a_1, a_n, d, n, S_n are called the **Elements** of the progression.

6. The common difference may be either positive or negative.

If d be *positive*, each term is greater than the preceding, and the series is called a *rising*, or an *increasing* progression.

E.g., $1 + 2 + 3 + 4 + \cdots$, wherein $d = 1$.

If d be negative, each term is less than the preceding, and the series is called a *falling*, or a *decreasing* progression.

E.g., $1 - 1 - 3 - 5 - \cdots$, wherein $d = -2$.

The nth Term of an Arithmetical Progression.

7. By the definition of an arithmetical progression,

$$a_1 = a_1,\ a_2 = a_1 + d,\ a_3 = a_2 + d = a_1 + 2\,d, \text{ etc.}$$

The law expressed by the formulæ for these first three terms is evidently general, and since the coefficient of d in each is one less than the number of the corresponding term, we have

$$a_n = a_1 + (n - 1)d. \qquad \text{(I.)}$$

That is, to find the nth term of an arithmetical progression: *Multiply the common difference by* $n - 1$, *and add the product to the first term.*

8. Ex. 1. Find the 15th term of the progression,

$$1 + 3 + 5 + 7 + \cdots$$

We have $a_1 = 1,\ d = 2,\ n = 15;$

therefore $a_{15} = 1 + (15 - 1)2 = 1 + 28 = 29.$

This formula may be used not only to find a_n, when a_1, d, and n are given, but also to find any one of the four numbers involved when the other three are given.

Ex. 2. If $a_5 = 3 (n = 5)$, and $a_1 = 1$, we have $3 = 1 + 4d$; whence $$d = \tfrac{1}{2}.$$

The Sum of *n* Terms of an Arithmetical Progression.

9. The successive terms in an arithmetical progression, from the first to the nth inclusive, may be obtained either by repeated additions of the common difference beginning with the first term, or by repeated subtractions of the common difference beginning with the nth term. We may therefore express the sum of n terms in two equivalent ways:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + \overline{n-2} \cdot d) + (a_1 + \overline{n-1} \cdot d),$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - \overline{n-2} \cdot d) + (a_n - \overline{n-1} \cdot d).$$

Whence, by addition,

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n),$$

wherein there are n binomials, $a_1 + a_n$.

Therefore, $2S_n = n(a_1 + a_n)$, or $\boldsymbol{S_n = \frac{n}{2}(a_1 + a_n)}$. (II.)

10. If the value of a_n, given in (I.), be substituted for a_1 in (II.), we obtain

$$\boldsymbol{S_n = \frac{n}{2}[2a_1 + (n-1)d]}. \qquad \text{(III.)}$$

Formula (II.) is used when a_1, a_n, and n are given; and (III.) when a_1, d, and n are given.

11. Ex. 1. If $a_1 = 1$, $a_5 = 3$, then $S_5 = \frac{5}{2}(1 + 3) = 10$.

Ex. 2. If $a_1 = -4$, $d = 2$, $n = 12$,

then $$S_{12} = \tfrac{12}{2}[2(-4) + 11 \times 2] = 84.$$

Either (II.) or (III.) can be used to determine any one of the five elements a_1, a_n, d, n, S_n, when the three others involved in the formula are known.

Ex. 3. Given $a_1 = -3$, $d = 2$, $S_n = 12$, to find n.

From (III.), $12 = \frac{n}{2}[-6 + 2(n-1)]$,

or $$n^2 - 4n = 12; \text{ whence } n = 6 \text{ and } -2.$$

The result 6 gives the series $-3-1+1+3+5+7, =12$.

Since the number of terms must be positive, the negative result, -2, is not admissible. But its meaning may be assumed to be that two terms, beginning with the last and counting toward the first, are to be taken.

12. Formulæ (I.) and (II.), or (I.) and (III.), may be used simultaneously to determine any two of the five numbers a_1, a_n, d, S_n, n when the three others are given.

Ex. Given $d=-2$, $a_n=-16$, $S_n=-60$, to find a_1 and n.

From (I.),
$$-16=a_1-2(n-1), \tag{1}$$
and from (II.),
$$-60=\frac{n}{2}(a_1-16). \tag{2}$$

Solving (1) and (2), we obtain $n=12$, $a_1=6$; and $n=5$, $a_1=-8$.

The two series are:

$$6+4+2+0-2-4-6-8-10-12-14-16,$$
and
$$-8-10-12-14-16,$$
both of which have $d=-2$, $a_n=-16$, $S_n=-60$.

Notice that in this example the sum of the terms which are not common to the two series is 0.

EXERCISES I.

Find the last term, and the sum of the terms, of each of the following arithmetical progressions:

1. $2+6+\cdots$ to 10 terms.
2. $3+1-\cdots$ to 13 terms.
3. $-5-2+\cdots$ to 21 terms.
4. $3+1\frac{1}{2}+\cdots$ to 40 terms.
5. $4+1\frac{3}{4}-\cdots$ to 31 terms.
6. $9+11+\cdots$ to n terms.
7. $n+2n+\cdots$ to 16 terms, to m terms.
8. $a+(a+b)+\cdots$ to 20 terms, to n terms.
9. $(m+2)+(4m+5)+\cdots$ to 40 terms, to n terms.
10. $\frac{a-1}{a}+\frac{a-3}{a}+\cdots$ to 30 terms, to n terms.

In each of the following arithmetical progressions find the values of the two elements not given:

11. $a_1 = 4,\ d = 5,\ n = 10.$ **12.** $a_n = 16,\ d = 2,\ n = 9.$

13. $a_1 = 2\frac{3}{5},\ n = 5,\ a_n = -1.9.$ **14.** $d = -4.8,\ n = 3,\ S_n = 28.5.$

15. $a_n = 13,\ n = 8,\ S_n = 100.$ **16.** $a_n = 2\frac{1}{6},\ n = 12,\ S_n = -7.$

17. $a_1 = 9,\ d = -1,\ a_n = 6.$ **18.** $a_1 = 22\frac{1}{3},\ a_n = -19\frac{2}{3},\ S_n = 20.$

19. $a_1 = 2,\ d = 5,\ S_n = 245.$ **20.** $a_n = 56,\ d = 5,\ S_n = 324.$

Arithmetical Means.

13. The **Arithmetical Mean** between two numbers is a third number, in value between the two, which forms with them an arithmetical progression.

E.g., 2 is an arithmetical mean between 1 and 3.

Let A stand for the arithmetical mean between a and b; then, by the definition of an arithmetical progression,

$$A - a = b - A,$$

whence

$$A = \frac{a + b}{2}.$$

That is, the arithmetical mean between two numbers is half their sum.

14. Arithmetical Means between two numbers are numbers, in value between the two, which form with them an arithmetical progression.

E.g., 2, 3, and 4 are three arithmetical means between 1 and 5.

Ex. Insert four arithmetical means between -2 and 9.

We have $n = 6,\ a_1 = -2,\ a_6 = 9.$

From (I.), $9 = -2 + 5d$, whence $d = \frac{11}{5}$.

The required means are $\frac{1}{5}$, $\frac{12}{5}$, $\frac{23}{5}$, $\frac{34}{5}$.

EXERCISES II.

Insert an arithmetical mean between

1. 45 and 31. **2.** $17\frac{1}{2}$ and $14\frac{1}{2}$. **3.** $2a$ and $-2b$.

4. $\dfrac{a-b}{a+b}$ and $\dfrac{a+b}{a-b}$. **5.** $\dfrac{x+1}{x-1}$ and $-\dfrac{x^3+1}{x^3-1}$.

6. Insert six arithmetical means between 7 and 35.

7. Insert twelve arithmetical means between 37 and -28.

8. Insert nine arithmetical means between $\frac{1}{5}$ and 12.

9. Insert twenty arithmetical means between -16 and 26.

10. Insert six arithmetical means between $a+b$ and $8a-13b$.

Problems.

15. Pr. Find the sum of all the numbers of three digits which are multiples of 7.

The numbers of three digits which are multiples of 7 are

$$7 \times 15,\ 7 \times 16,\ 7 \times 17,\ \cdots,\ 7 \times 142.$$

Their sum is $\qquad 7(15 + 16 + \cdots + 142).$

The series within the parentheses is an arithmetical progression, in which $a_1 = 15$, $d = 1$, $n = 128$, and $a_{128} = 142$.

Therefore $\qquad S_{128} = 10048.$

The required sum is therefore $7 \times 10048, = 70336$.

16. In many examples the elements necessary for determining the required element or elements directly from (I.)–(III.) are not given, but in their place equivalent data.

Ex. 1. The sixth term of an A. P. is 17, and the eleventh term is 32. Find the first term and the common difference.

We have $\qquad a_6 = 17,\ a_{11} = 32.$

From (I.), $\quad 17 = a_1 + 5d$, and $32 = a_1 + 10d$.

Solving these equations, $a_1 = 2$, $d = 3$.

Or we could have regarded 17 as the first term and 32 as the last term of a progression of six terms. Then, by (I.), $32 = 17 + 5d$, whence $d = 3$.

By (I.) again, $17 = a_1 + 5 \times 3$; whence $a_1 = 2$, as above.

EXERCISES III.

1. Find the sixth term, and the sum of eleven terms, of an A. P. whose eighth term is 11 and whose fourth term is -1.

2. The sixteenth term of an A. P. is -5, and the forty-first term is 45. What is the first term, and the sum of twenty terms?

3. Find the sum of all the even numbers from 2 to 50 inclusive.

4. Find the sum of thirty consecutive odd numbers, of which the last is 127.

5. The sum of the eighth and fourth terms of an A. P. of twenty terms is 24, and the sum of the fifteenth and nineteenth terms is 68. What are the elements of the progression?

6. The sum of the second and twentieth terms of an A. P. is 10, and their product is $23\frac{47}{64}$. What is the sum of sixteen terms?

7. The sixth term of an A. P. is 30, and the sum of the first thirteen terms is 455. What is the sum of the first thirty terms?

8. What value of x will make the arithmetical mean between $x^{\frac{1}{2}}$ and $x^{\frac{1}{4}}$ equal to 6?

9. Find the sum of all even numbers of two digits.

10. How many consecutive odd numbers beginning with 7 must be taken to give a sum 775?

11. Insert between 0 and 6 a number of arithmetical means so that the sum of the terms of the resulting A. P. shall be 39.

12. Find the number of arithmetical means between 1 and 19, if the first mean is to the last mean as 1 to 7.

13. The sum of the terms of an A. P. of six terms is 66, and the sum of the squares of the terms is 1006. What are the elements of the progression?

14. The sum of the terms of an A. P. of twelve terms is 354, and the sum of the even terms is to the sum of the odd terms as 32 to 27. What is the common difference?

15. How many positive integers of three digits are there which are divisible by 9? Find their sum.

16. Show that the sum of $2n+1$ consecutive integers is divisible by $2n+1$.

17. Prove that if the same number be added to each term of an A. P., the resulting series will be an A. P.

18. Prove that if each term of an A. P. be multiplied by the same number, the resulting series will be an A. P.

19. Prove that if in the equation $y = ax + b$, we substitute c, $c + d$, $c + 2d$, $\cdots$, in turn for x, the resulting values of y will form an A. P.

20. A laborer agreed to dig a well on the following conditions: for the first yard he was to receive \$2, for the second \$2.50, for the third \$3, and so on. If he received \$42.50 for his work, how deep was the well?

21. On a certain day the temperature rose $\frac{1}{2}°$ hourly from 5 to 11 A.M., and the average temperature for that period was 8°. What was the temperature at 8 A.M.?

22. Twenty-five trees are planted in a straight line at intervals of 5 feet. To water them, the gardener must bring water for each tree separately from a well which is 10 feet from the first tree and in line with the trees. How far has the gardener walked when he has watered all the trees?

GEOMETRICAL PROGRESSION.

17. A **Geometrical Series,** or, as it is more commonly called, a **Geometrical Progression** (G. P.), is a series in which each term after the first is formed by multiplying the preceding term by a constant number. See Art. 1, (2).

18. Evidently this definition is equivalent to the statement that the ratio of any term to the preceding is constant.

For this reason the constant multiplier of the first definition is called the **Ratio** of the progression.

19. Let a_1 stand for the first term of the series,
a_n for the nth (*any*) term,
r for the ratio,
and S_n for the sum of n terms.

20. The ratio may be either larger or smaller than 1; in the former case the progression is called a *rising* or *ascending* progression; in the latter a *falling* or *descending* progression.

E.g., $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$, in which $r = \frac{3}{2}$,

and $\frac{1}{2} - 1 + 2 - 4 + 8 \cdots$, in which $r = -2$,

are ascending progressions; while

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, in which $r = \frac{1}{2}$,

and $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$, in which $r = -\frac{2}{3}$,

are *descending* progressions.

The *n*th Term of a Geometrical Progression.

21. By the definition of a geometrical progression

$$a_1 = a_1,\ a_2 = a_1 r,\ a_3 = a_2 r = a_1 r^2,\ a_4 = a_3 r = a_1 r^3, \text{ etc.}$$

The law expressed by the relations for these first four terms is evidently general, and since the exponent of r in each is one less than the number of the corresponding term, we have

$$\boldsymbol{a_n = a_1 r^{n-1}}. \qquad \text{(I.)}$$

That is, to find the nth term of a geometrical progression: *Raise the ratio to a power one less than the number of the term, and multiply the result by the first term.*

Ex. 1. If $a_1 = \frac{1}{2}$, $r = 3$, $n = 5$, then $a_5 = \frac{1}{2} \cdot 3^4 = \frac{81}{2}$.

This relation may also be used to find not only a_n, when a_1, r, and n are given, but also to find the value of any one of the four numbers when the other three are given.

Ex. 2. If $a_1 = 4$, $a_6 = \frac{1}{8}$, $n = 6$, then $\frac{1}{8} = 4r^5$, whence $r = \frac{1}{2}$.

The Sum of a Geometrical Progression.

22. We have $S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$, (1)

and $rS_n = \quad a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1} + a_1 r^n$. (2)

Consequently, subtracting (2) from (1),

$$S_n(1 - r) = a_1 - a_1 r^n,$$

whence

$$\boldsymbol{S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1(r^n - 1)}{r - 1}}. \qquad \text{(II.)}$$

Substituting a_n for a_1r^{n-1} in (II.), we have

$$S_n = \frac{a_1 - a_nr}{1 - r} = \frac{a_nr - a_1}{r - 1}. \qquad \text{(III.)}$$

The first forms of (II.) and (III.) are to be used when $r < 1$, the second when $r > 1$.

23. Ex. 1. Given $a_1 = 3,\ r = 2,\ n = 6,$ to find S_6.

From (II.), $\quad S_6 = \dfrac{3\,(2^6 - 1)}{2 - 1} = 189.$

Formulæ (II.) and (III.) may be used not only to find S_n when a_1, r, and n, or a_1, a_n, and r are given, but also to find the value of any one of the four numbers when the other three are given.

Ex. 2. Given $S_n = -63\frac{1}{2},\ a_1 = -\frac{1}{2},\ a_n = -32,$ to find r.

By (III.), $\; -63\frac{1}{2} = \dfrac{-\frac{1}{2} + 32r}{1 - r}$, whence $r = 2$.

24. Formulæ (I.) and (II.), or (I.) and (III.), may be used simultaneously to determine any two of the five elements, a_1, a_n, r, S_n, n, when the three other elements are given.

Ex. Given $r = 2,\ a_n = 16,\ S_n = 31\frac{1}{2},$ to find a_1 and n.

From (III.), $31\frac{1}{2} = \dfrac{16 \times 2 - a_1}{2 - 1}$, whence $a_1 = \frac{1}{2}$.

From (I.), $\quad 16 = \frac{1}{2} \cdot 2^{n-1}$, whence $n = 6$.

EXERCISES IV.

Find the last term and the sum of the terms of each of the following geometrical progressions:

1. $3 + 6 + \cdots$ to six terms.
2. $2 - 4 + \cdots$ to ten terms.
3. $32 - 16 + \cdots$ to seven terms.
4. $1\frac{3}{8} + 2\frac{3}{8} + \cdots$ to six terms.
5. $2 - 2^2 + \cdots$ to eleven terms.
6. $\dfrac{2}{\sqrt{2}} + \dfrac{1}{2} + \cdots$ to n terms.
7. $1 + (1 + a) + \cdots$ to four terms, to n terms.

In each of the following geometrical progressions find the values of the elements not given:

8. $a_1 = 1,\ r = 4,\ n = 5.$ **9.** $a_n = 10,\ r = 2,\ n = 4.$

10. $a_n = 96,\ n = 4,\ S_n = 127.5.$ **11.** $r = 10,\ n = 7,\ S_n = 3{,}333{,}333.$

12. $a_1 = 74\frac{2}{3},\ n = 6,\ a_n = 2\frac{1}{3}.$ **13.** $a_1 = 7,\ r = 10,\ a_n = 700.$

14. $a_1 = 1,\ a_n = 512,\ S_n = 1023.$ **15.** $a_n = 3125,\ r = 5,\ S_n = 3905.$

16. $a_1 = 4,\ r = 3,\ S_n = 118{,}096.$ **17.** $a_1 = 100,\ n = 3,\ S_n = 700.$

25. The Sum of an Infinite Geometrical Progression. — If the number of terms in a geometrical progression is unlimited, the exact value of the sum of the series cannot be obtained. Thus, in the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots \text{ without end,}$$

the sum continually increases as more and more terms are included in it.

We have
$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = \frac{1}{1 - \frac{1}{2}} - \frac{(\frac{1}{2})^n}{\frac{1}{2}}$$
$$= 2 - (\tfrac{1}{2})^{n-1}.$$

And
$$S_1 = 1,\ S_2 = 1\tfrac{1}{2},\ S_3 = 1\tfrac{3}{4},\ S_4 = 1\tfrac{7}{8},\ \cdots$$
$$S_{1000} = 2 - (\tfrac{1}{2})^{999};\ \text{and so on.}$$

We thus see that, although the sum of this series grows larger and larger, it does not increase without limit, but approaches the value 2 more and more nearly as more and more terms are included in the sum. Evidently the sum can be made to differ from 2 by as little as we please, by taking a sufficient number of terms.

We therefore call 2 *the limit of the sum* of the series, or more briefly, the *sum* of the series. The *exact* sum 2, however, can never be obtained.

26. In general, when $r < 1$, the term a_1r^n in the formula

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$

decreases as n increases. It can be proved, as in the particular example, that this term can be made as small as we please, by taking n sufficiently great.

Therefore, when $r < 1$, we take

$$S = \frac{a_1}{1 - r}$$

as the sum of the infinite geometrical progression.

This theory can be applied to find the value of a repeating (recurring) decimal.

Ex. Verify that $\qquad .\dot{6} = \frac{2}{3}.$

We have $\qquad .666 \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots,$

a geometrical progression whose first term is $\frac{6}{10}$ and whose ratio is $\frac{1}{10}$. Consequently

$$S = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{9} = \frac{2}{3}.$$

EXERCISES V.

Find the sum of the following infinite geometrical progressions:

1. $6 + 4 + \cdots$. 2. $60 + 15 + \cdots$. 3. $10 - 6 + \cdots$.

4. $\frac{1}{2} + \frac{1}{4} + \cdots$. 5. $1 - \frac{1}{3} + \cdots$. 6. $5 - \frac{1}{2} + \cdots$.

7. $\frac{3}{2} - \frac{2}{3} + \cdots$. 8. $\sqrt{\frac{3}{2}} + \sqrt{\frac{2}{3}} + \cdots$. 9. $\sqrt{.2} + \sqrt{\frac{1}{125}} + \cdots$.

10. $1 + x + x^2 + \cdots$, when $x < 1$.

11. $1 + \frac{1}{x} + \frac{1}{x^2} + \cdots$, when $x > 1$.

Find the value of each of the following repeating decimals:

12. $.44 \cdots$. 13. $.99 \cdots$. 14. $.2727 \cdots$.

15. $.015015 \cdots$. 16. $.199 \cdots$. 17. $1.0909 \cdots$.

18. $.122323 \cdots$. 19. $.201475475 \cdots$.

Verify each of the following identities:

20. $\sqrt{.44 \cdots} = .66 \cdots$. 21. $\sqrt{.6944 \cdots} = .833 \cdots$.

Geometrical Means.

27. A **Geometrical Mean** between two numbers is a number, in value between the two, which forms with them a geometrical progression.

E.g., $+2$, or -2, is a geometrical mean between 1 and 4.

Let G be the geometrical mean between a and b.

Then by definition of a geometrical progression,

$$\frac{G}{a}=\frac{b}{G}; \text{ whence } \boldsymbol{G=\pm\sqrt{(ab)}}.$$

That is, the geometrical mean between two numbers is the square root of their product.

Ex. Find the geometrical mean between 1 and $\frac{4}{9}$. We have

$$G=\pm\sqrt{(1\times\tfrac{4}{9})}=\pm\tfrac{2}{3}.$$

28. **Geometrical Means** between two numbers are numbers, in value between the two, which form with them a geometrical progression. *E.g.*, 4 and 16 are two geometrical means between 1 and 64; and 2, 4, 8, 16, 32 are five geometrical means between 1 and 64.

Ex. Insert five geometrical means between 1 and 729.

We have $\quad a_1=1,\ n=7,\ a_n=729.$

Therefore $\quad 729=r^6$, or $r=\pm 3$.

The required means are:

$$\pm 3,\ 9,\ \pm 27,\ 81,\ \pm 243.$$

EXERCISES VI.

Insert a geometrical mean between

1. 2 and 8. 2. 12 and 3. 3. $\frac{1}{8}$ and $\frac{1}{125}$.

4. $\sqrt{a}$ and $\sqrt{(2a)}$. 5. $75m^3$ and $3mn^4$. 6. $\dfrac{p}{q}$ and $\dfrac{q}{p}$.

7. $(a-b)^2$ and $(a+b)^2$. 8. $(a^2+1)(a^2-1)^{-1}$ and $\frac{1}{4}(a^4-1)$.

9. Insert five geometrical means between 2 and 1458.

10. Insert seven geometrical means between 2 and 512.

11. Insert six geometrical means between 3 and -384.

12. Insert six geometrical means between 5 and -640.

13. Insert nine geometrical means between 1 and $\frac{1024}{59049}$.

Problems.

29. Pr. A farmer agrees to sell 12 sheep on the following terms: he is to receive 2 cents for the first sheep, 4 cents for the second, 8 cents for the third, and so on. How much does he receive for the twelfth sheep, and how much for the 12 sheep, and what is the average price?

We have $a_1 = 2,\ n = 12,\ r = 2.$

Then $a_{12} = 2 \times 2^{11} = 2^{12} = 4096.$

And $$S_{12} = \frac{2(2^{12}-1)}{2-1} = 2 \times 4095 = 8190.$$

That is, he receives 4096 cents, or \$ 40.96, for the twelfth sheep, and 8190 cents, or \$ 81.90, for the 12 sheep.

The average price is $\frac{81.90}{12}$, = \$ 6.82½.

30. In many examples the elements necessary for determining the element or elements directly from (I.)–(III.) are not given, but in their place equivalent data.

Ex. The fifth term of a G. P. is 48, and the eighth term is 384. Find the first term and the ratio.

From (I.), $48 = a_1r^4$, and $384 = a_1r^7$;

whence $r^3 = 8$, or $r = 2$. Therefore $a_1 = 3$.

Or, we could have regarded 48 as the first term and 384 as the last term of a progression of four terms. Then by (I.), $384 = 48\,r^3$, whence $r = 2$ as before.

EXERCISES VII.

1. The first term of a G. P. of six terms is 768, and the last term is one-sixteenth of the fourth term. What is the sum of the six terms of the progression?

2. The first term of a G. P. of ten terms is 3, and the sum of the first three terms is one-eighth of the sum of the next three terms. Find the elements of the progression.

3. The twelfth term of a G. P. is 1536, and the fourth term is 6. What is the ratio, and the sum of the first eleven terms?

4. In a G. P. of eight terms, the sum of the first seven terms is $444\frac{1}{2}$, and is to the sum of the last seven terms as 1 to 2. Find the elements of the progression.

5. The sum of the first four terms of a G. P. is 15, and the sum of the terms from the second to the fifth inclusive is 30. What is the first term, and the ratio?

6. Find the elements of a G. P. of six terms whose first term is 1, and the sum of whose first six terms is 28 times the sum of the first three terms.

7. The sum of the first three terms of a G. P. is 21, and the sum of their squares is 189. What is the first term?

8. The product of the first three terms of a G. P. is 216, and the sum of their cubes is 1971. What is the first term, and the ratio?

9. If the numbers 1, 1, 3, 9 be added to the first four terms of an A. P., respectively, the resulting terms will form a G. P. What is the first term, and the common difference of the A. P.?

10. A G. P. and an A. P. have a common first term 3, the difference between their second terms is 6, and their third terms are equal. What is the ratio of the G. P., and the common difference of the A. P.?

11. Show that, if all the terms of a G. P. be multiplied by the same number, the resulting series will form a G. P.

12. Show that the series whose terms are the reciprocals of the terms of a G. P. is a G. P.

13. Show that the product of the first and last terms of a G. P. is equal to the product of any two terms which are *equally* distant from the first and last terms respectively.

14. A merchant's investment yields him each year after the first, three times as much as the preceding year. If his investment paid him \$ 9720 in four years, how much did he realize the first year and the fourth year?

15. Given a square whose side is $2a$. The middle points of its adjacent sides are joined by lines forming a second square inscribed in the first. In the same manner a third square is inscribed in the second, a fourth in the third, and so on indefinitely. Find the sum of the perimeters of all the squares.

HARMONICAL PROGRESSION.

31. A **Harmonical Progression** (H. P.) is a series the reciprocals of whose terms form an arithmetical progression.

E.g.,
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$
is a harmonical progression, since
$$1 + 2 + 3 + 4 + \cdots$$
is an arithmetical progression.

Consequently to every harmonical progression there corresponds an arithmetical progression, and *vice versa.*

32. Any term of a harmonical progression is obtained by finding the same term of the corresponding arithmetical progression and taking its reciprocal.

Ex. Find the eleventh term of the harmonical progression $4, 2, \frac{4}{3}, \cdots$.

The corresponding arithmetical progression is
$$\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \cdots,$$
and its eleventh term is $\frac{11}{4}$.

Therefore the eleventh term of the given progression is $\frac{4}{11}$.

33. No formula has been derived for the sum of n terms of a harmonical progression.

34. A **Harmonical Mean** between two numbers is a number in value between the two, which forms with them a harmonical progression.

E.g., $\frac{3}{2}$ is a harmonical mean between $\frac{1}{2}$ and $-\frac{3}{2}$.

Let H stand for the harmonical mean between a and b, then $\frac{1}{H}$ is an arithmetical mean between $\frac{1}{a}$ and $\frac{1}{b}$. Consequently

$$\frac{1}{H}=\frac{\frac{1}{a}+\frac{1}{b}}{2}, \text{ or } H=\frac{2\,ab}{a+b}.$$

Ex. Insert a harmonical mean between 2 and 5.

We have $$H=\frac{2\times 2\times 5}{2+5}=\frac{20}{7}.$$

35. Harmonical Means between two numbers are numbers, in value between the two, which form with them a harmonical progression.

E.g., $\frac{3}{2}$, 1, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{1}{2}$ are five harmonical means between 3 and $\frac{3}{7}$.

Ex. Insert four harmonical means between 1 and 10.

We have first to insert four arithmetical means between 1 and $\frac{1}{10}$, and obtain

$$\tfrac{41}{50},\ \tfrac{32}{50},\ \tfrac{23}{50},\ \tfrac{14}{50}.$$

The required harmonical means are therefore

$$\tfrac{50}{41},\ \tfrac{50}{32},\ \tfrac{50}{23},\ \tfrac{50}{14}.$$

Problems.

36. Pr. 1. The geometrical mean between two numbers is $\frac{1}{2}$, and the harmonical mean is $\frac{2}{5}$. What are the numbers?

Let x and y represent the two numbers.

Then $$\sqrt{(xy)}=\tfrac{1}{2}, \text{ or } xy=\tfrac{1}{4}; \tag{1}$$

and $$\frac{2\,xy}{x+y}=\frac{2}{5}, \text{ or } 5\,xy=x+y. \tag{2}$$

Solving (1) and (2), we obtain $x=1$, $y=\frac{1}{4}$, and $x=\frac{1}{4}$, $y=1$.

EXERCISES VIII.

Find the last term of each of the following harmonical progressions:

1. $1+\frac{2}{3}+\frac{1}{2}+\cdots$ to 8 terms.
2. $\frac{1}{3}+\frac{1}{8}+\frac{1}{13}+\cdots$ to 15 terms.
3. $2-2-\frac{2}{3}-\cdots$ to 11 terms.
4. $-8-\frac{8}{9}-\frac{8}{17}-\cdots$ to 16 terms.

5. $\dfrac{1}{a}+\dfrac{1}{2a}+\dfrac{1}{3a}+\cdots$ to 25 terms.

6. $\dfrac{1}{\sqrt{2}}+\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{2+\sqrt{2}}+\cdots$ to 30 terms.

Find the harmonical mean between

7. 2 and 4. 8. -3 and 4. 9. $\frac{1}{7}$ and $\frac{1}{9}$.

10. $\dfrac{1}{x-1}$ and $-\dfrac{1}{x+1}$. 11. $\dfrac{a-b}{a+b}$ and $\dfrac{a+b}{a-b}$.

12. Insert 5 harmonical means between 5 and $\frac{1}{5}$.

13. Insert 10 harmonical means between 3 and $\frac{1}{4}$.

14. Insert 4 harmonical means between -7 and $\frac{1}{2}$.

15. If b be the harmonical mean between a and c, prove that

$$\frac{a-b}{b-c}=\frac{a}{c}.$$

16. The arithmetical mean between two numbers is 6, and the harmonical mean is $\frac{35}{6}$. What are the numbers?

17. If one number exceeds another by two, and if the arithmetical mean exceeds the harmonical mean by $\frac{1}{10}$, what are the numbers?

18. The seventh term of a harmonical progression is $\frac{1}{15}$, and the twelfth term is $\frac{1}{25}$. What is the twentieth term?

19. The tenth term of a harmonical progression is $\frac{1}{5}$, and the twentieth term is $\frac{1}{10}$. What is the first term?

CHAPTER XXII.

THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS.

1. The expansions of the powers of a binomial, from the third to the fourth inclusive, were given in Ch. XIII., Arts. 7–8, and the laws governing the expansion of these powers were stated.

As yet, however, we cannot infer that these laws hold for the fifth power without multiplying the expansion of the fourth power by $a+b$; nor for the sixth power without next multiplying the expansion of the fifth power by $a+b$; and so on.

If, however, we prove that, provided the laws hold for any particular power, they hold for the next higher power, we can infer, without further proof, that because the laws hold for the fourth power, they hold also for the fifth; then that because they hold for the fifth, they hold also for the sixth, and so on to any higher power.

2. If the laws (i.)–(vi.) hold for the rth power, we have

$$(a+b)^r=a^r+ra^{r-1}b+\frac{r(r-1)}{1\cdot 2}a^{r-2}b^2+\frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}a^{r-3}b^3+\cdots.$$

Notice that only the first four terms of the expansion are written. But it is often necessary to write any term (the kth, say) without having written all the preceding terms.

To derive this term, observe that the following laws hold for each term of the expansion:

(i.) *The exponent of b is one less than the number of the term (counting from the left).*

Thus in the first term we have $b^{1-1}=b^0=1$; in the second, $b^{2-1}=b$; in the tenth, $b^{10-1}=b^9$; and in the kth term, b^{k-1}.

(ii.) *The exponent of a is equal to the binomial exponent less the exponent of b.*

Thus, in the first term we have $a^{r-0}=a^r$; in the second, a^{r-1}; in the tenth, a^{r-9}; and in the kth term, $a^{r-(k-1)}$, $= a^{r-k+1}$.

(iii.) *The number of factors (beginning with* 1 *and increasing by* 1) *in the denominator of each coefficient, and the number of factors (beginning with r and decreasing by* 1) *in the numerator of each coefficient, is equal to the exponent of b in that term.*

Thus, in the coefficient of the second term the denominator is 1 and the numerator is r; in that of the third term the denominator is $1\cdot 2$ and the numerator is $r(r-1)$; in the tenth term the denominator is $1\cdot 2 \cdots 9$ and the numerator is $r(r-1)\cdots(r-8)$; and in the kth term the denominator is $1\cdot 2\cdot 3\cdots(k-1)$, and the numerator is

$$r(r-1)\cdots[r-(k-2)], = r(r-1)\cdots(r-k+2).$$

Therefore the kth term in the expansion of $(a+b)^r$ is

$$\frac{r(r-1)(r-2)\cdots(r-k+2)}{1\cdot 2\cdot 3\cdots(k-1)}a^{r-k+1}b^{k-1}.$$

In like manner, any other term can be written.

Thus, the $(k-1)$th term is

$$\frac{r(r-1)(r-2)\cdots(r-k+3)}{1\cdot 2\cdot 3\cdots(k-2)}a^{r-k+2}b^{k-2}.$$

3. We can now prove that, if the laws (i.)–(vi.) hold for $(a+b)^r$, they also hold for $(a+b)^{r+1}$; that is, if they hold for any power they hold for the next higher power. Assuming, then, that the laws hold for $(a+b)^r$, we have

$$\begin{aligned}(a+b)^r = a^r + ra^{r-1}b + \frac{r(r-1)}{1\cdot 2}a^{r-2}b^2 + \cdots \\ + \frac{r(r-1)(r-2)\cdots(r-k+3)}{1\cdot 2\cdot 3\cdots(k-2)}a^{r-k+2}b^{k-2} \\ + \frac{r(r-1)(r-2)\cdots(r-k+3)(r-k+2)}{1\cdot 2\cdot 3\cdots(k-2)(k-1)}a^{r-k+1}b^{k-1} + \cdots.\end{aligned}$$

The first three terms of the expansion are written, then all terms are omitted, except the $(k-1)$th and the kth.

Multiplying the expansion of $(a+b)^r$ by $(a+b)$, we obtain:

$$(a+b)^{r+1} = a^{r+1} + ra^rb + \frac{r(r-1)}{1\cdot 2}a^{r-1}b^2 + \cdots$$

$$+\frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)}a^{r-k+2}b^{k-1} + \cdots$$

$$+ a^rb + ra^{r-1}b^2 + \cdots + \frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)}a^{r-k+2}b^{k-1} + \cdots$$

$$= a^{r+1} + (r+1)a^rb + \left[\frac{r(r-1)}{1\cdot 2} + r\right]a^{r-1}b^2 + \cdots$$

$$+\left[\frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)} + \frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)}\right]a^{r-k+2}b^{k-1} + \cdots.$$

But
$$\frac{r(r-1)}{1\cdot 2} + r = \frac{r^2 - r + 2r}{1\cdot 2} = \frac{(r+1)r}{1\cdot 2};$$

and
$$\frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)} + \frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)}$$

$$= \frac{r(r-1)\cdots(r-k+2) + r(r-1)\cdots(r-k+3)(k-1)}{1\cdot 2\cdots(k-1)}$$

$$= \frac{r(r-1)\cdots(r-k+3)(r-k+2+k-1)}{1\cdot 2\cdots(k-1)}$$

$$= \frac{(r+1)r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-1)}.$$

Therefore,

$$(a+b)^{r+1} = a^{r+1} + (r+1)a^rb + \frac{(r+1)r}{1\cdot 2}a^{r-1}b^2 + \cdots$$

$$+ \frac{(r+1)r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-1)}a^{r-k+2}b^{k-1} + \cdots.$$

The laws (i.)–(vi.) hold for the above expansion of $(a+b)^{r+1}$. We therefore conclude that if the expansion holds for $(a+b)^r$, it also holds for $(a+b)^{r+1}$.

Consequently, since the expansion holds for the fourth power, it holds for the fifth, and so on to any positive integral power.

The method of proof employed in this article is called **Proof by Mathematical Induction.**

4. We may now write the expansion of $(a+b)^n$, wherein n is any positive integer:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \cdots.$$

In particular, if $a = 1$, and $b = x$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \cdots.$$

5. The expansion of $(a-b)^n$ can be at once written from that of $(a+b)^n$.

We have $\quad (a-b)^n = [a+(-b)]^n$

$$= a^n + na^{n-1}(-b) + \frac{n(n-1)}{1 \cdot 2}a^{n-2}(-b)^2 + \cdots$$

$$= a^n - na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 - \cdots.$$

Observe that the signs of the terms alternate, + and −, beginning with the first, or that the terms containing *even* powers of b are *positive*, and those containing *odd* powers of b are *negative*.

6. When n is a positive integer, the number of terms in the expansion is limited.

E.g., $(a+b)^5 = a^5 + 5\,a^4b + \dfrac{5 \cdot 4}{1 \cdot 2}a^3b^2 + \dfrac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}a^2b^3$

$$+ \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}ab^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}b^5$$

$$+ \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}a^{-1}b^6 + \cdots.$$

The coefficients of the seventh and all succeeding terms contain 0 as a factor. Therefore these terms drop out, and the expansion ends with the sixth term. In general, the expansion of $(a+b)^n$ ends with the $(n+1)$th term. For, the coefficients of the $(n+2)$th and all succeeding terms contain $n-n$, or 0, as a factor.

7. The expansion of $(a+b)^n$ may also be written in descending powers of b.

Thus, $(b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{1\cdot 2}b^{n-2}a^2 + \cdots,$

wherein b^n is the last term of the expansion given in Art. 4, n the coefficient of the next to the last term, and so on.

We therefore conclude:

In the expansion of $(a+b)^n$, wherein n is a positive integer, the coefficients of terms equally distant from the beginning and end of the expansion are equal.

8. In Exs. 1–2 which follow, the coefficients are computed by the principle given in Ch. XIII., Art. 7 (v.).

Ex. 1. Expand $(1-2x^2)^5$.

We have
$$\begin{aligned}(1-2x^2)^5 &= 1^5 - 5\cdot 1^4\cdot(2x^2) + 10\cdot 1^3\cdot(2x^2)^2 \\ &\quad - 10\cdot 1^2\cdot(2x^2)^3 + 5\cdot 1\cdot(2x^2)^4 - (2x^2)^5 \\ &= 1 - 10x^2 + 40x^4 - 80x^6 + 80x^8 - 32x^{10}.\end{aligned}$$

In expanding a binomial, the coefficients of the terms after the middle term may be at once written by the principle of the preceding article. This remark applies to the expansion before it is reduced, as in Ex. 1.

Ex. 2. Find the first five terms of $(a^{-\frac{1}{2}} + 2b^{-2})^{11}$.

We have
$$\begin{aligned}(a^{-\frac{1}{2}} + 2b^{-2})^{11} &= (a^{-\frac{1}{2}})^{11} + 11(a^{-\frac{1}{2}})^{10}(2b^{-2}) + 55(a^{-\frac{1}{2}})^9(2b^{-2})^2 \\ &\quad + 165(a^{-\frac{1}{2}})^8(2b^{-2})^3 + 330(a^{-\frac{1}{2}})^7(2b^{-2})^4 + \cdots \\ &= a^{-\frac{11}{2}} + 22a^{-5}b^{-2} + 220a^{-\frac{9}{2}}b^{-4} + 1320a^{-4}b^{-6} \\ &\quad + 5280a^{-\frac{7}{2}}b^{-8} + \cdots.\end{aligned}$$

9. Ex. Find the seventh term in $(2x-3y)^{11}$.

In the seventh term the exponent of $-3y(=b)$ is 6; the exponent of $2x(=a)$ is $11-6,\ =5$. The denominator of the coefficient contains six factors beginning with 1, and the numerator contains six factors beginning with 11. Therefore the seventh term is

$$\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(2x)^5(-3y)^6,\ = 10777536\,x^5y^6.$$

If the second term of the binomial is negative, it is better, in finding a particular term, to write the binomial in the form $[a+(-b)]$, as in the above example.

EXERCISES.

Write the expansion of each of the following powers:

1. $(a+b)^6$. **2.** $(x-y)^7$. **3.** $(a^2+b^2)^8$.

4. $(x^{-1}+y^3)^4$. **5.** $(a^{\frac{1}{2}}-b^4)^5$. **6.** $(x^{-2}+y^{\frac{1}{2}})^6$.

7. $(x^{\frac{1}{2}}-y^{\frac{2}{3}})^4$. **8.** $(a^{-3}+b^{-1})^5$. **9.** $(m^{-\frac{1}{4}}-n^{\frac{1}{2}})^6$.

10. $\left(x-\dfrac{a}{x}\right)^5$. **11.** $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)^6$. **12.** $\left(x+\dfrac{1}{x^2}\right)^7$.

13. $(a-5)^6$. **14.** $(2x+3y)^5$. **15.** $(4a^2-\frac{1}{5}b^{-\frac{1}{2}})^4$.

16. $(x^2-\sqrt{y})^4$. **17.** $(2\sqrt{a}-\frac{1}{3}\sqrt{b})^5$. **18.** $(x^3-y^2\sqrt{-3})^6$.

19. $\left(\sqrt{\dfrac{a}{n}}+\sqrt{\dfrac{n}{a}}\right)^9$. **20.** $\left(\dfrac{2}{\sqrt[3]{a^2}}-\dfrac{a\sqrt{a}}{2}\right)^5$. **21.** $\left(n^2+\dfrac{2a}{n^{-1}}\right)^6$.

22. $(\sqrt{-2}+2x^{-\frac{2}{7}})^7$. **23.** $(\sqrt[4]{a}+\sqrt[4]{b})^8$. **24.** $(a-\sqrt{-a})^8$.

25. $(ab^{-2}-b^2x)^9$. **26.** $(x^2-\sqrt{-x})^9$. **27.** $(a^2b+b^{-3})^{10}$.

28. $[\sqrt{(x+1)}-\sqrt{(x-1)}]^4$. **29.** $[\sqrt[3]{(a+b)}+\sqrt[3]{(a-b)}]^6$.

Simplify each of the following expressions:

30. $(1+\sqrt{-x})^8+(1-\sqrt{-x})^8$. **31.** $(x+\sqrt{-3})^9-(x-\sqrt{-3})^9$.

Write the expansion of each of the following powers:

32. $(1-x+x^2)^3$. **33.** $(2-3x+x^2)^4$.

34. $(1+a^{\frac{1}{2}}-a^{-2})^3$. **35.** $(1-x\sqrt{2}+x^2\sqrt{3})^4$.

Write the

36. 3d term of $(a+b)^{15}$. **37.** 5th term of $(a-b)^{16}$.

38. 6th term of $(a^{\frac{1}{10}}+b^{\frac{1}{5}})^{15}$. **39.** 7th term of $(a^n-a^{-n})^{14}$.

40. 6th term of $\left(\sqrt[3]{m}-\dfrac{2x}{\sqrt[3]{m^2}}\right)^{12}$. **41.** 15th term of $\left(a^3+\dfrac{1}{a}\right)^{20}$.

42. 12th term of $(x-\sqrt{-x})^{20}$. **43.** 9th term of $(\sqrt{x}-ax^{\frac{2}{3}})^{16}$.

44. Write the middle term of $(x\sqrt{x}-1)^4$.

45. Write the middle terms of $(a^{\frac{1}{3}}+x^{\frac{1}{2}})^9$.

CHAPTER XXIII.

PERMUTATIONS AND COMBINATIONS.

DEFINITIONS.

1. The following examples will illustrate the character of an important class of problems.

Pr. 1. Write the numbers of two figures each which can be formed from the three figures, 4, 5, 6.

We have 45, 54, 46, 64, 56, 65.

Pr. 2. What committees of two persons each can be appointed from the three persons, A, B, C?

The committees may consist of A, B; A, C; or B, C.

These problems make clear the difference between groups of things, selected from a given number of things, in which *the order is taken into account*, as in Pr. 1, and in which *the order is not taken into account*, as in Pr. 2.

2. We are thus naturally led to the following definitions:

A Permutation of any number of things is a group of some or all of them, *arranged in a definite order.*

A Combination of any number of things is a group of some or all of them, *without reference to order.*

3. It follows from these definitions that two permutations are different when some or all of the things in them are different, or when their order of arrangement is different; and that two combinations are different only when at least one thing in one is not contained in the other.

Thus, *ab* and *ba* are different permutations, but the same combination.

PERMUTATIONS.

4. The permutations of a, b, c, d are:

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
a\left\{\begin{array}{l} ab\left\{\begin{array}{l} abc \\ abd \end{array}\right. \\ ac\left\{\begin{array}{l} acb \\ acd \end{array}\right. \\ ad\left\{\begin{array}{l} adb \\ adc \end{array}\right. \end{array}\right. & & & \begin{array}{l} abcd \\ abdc \\ acbd \\ acdb \\ adbc \\ adcb \end{array}
\end{array}
\qquad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
b\left\{\begin{array}{l} ba\left\{\begin{array}{l} bac \\ bad \end{array}\right. \\ bc\left\{\begin{array}{l} bca \\ bcd \end{array}\right. \\ bd\left\{\begin{array}{l} bda \\ bdc \end{array}\right. \end{array}\right. & & & \begin{array}{l} bacd \\ badc \\ bcad \\ bcda \\ bdac \\ bdca \end{array}
\end{array}
$$

$$
c\left\{\begin{array}{l} ca\left\{\begin{array}{l} cab \\ cad \end{array}\right. \\ cb\left\{\begin{array}{l} cba \\ cbd \end{array}\right. \\ cd\left\{\begin{array}{l} cda \\ cdb \end{array}\right. \end{array}\right. \quad \begin{array}{l} cabd \\ cadb \\ cbad \\ cbda \\ cdab \\ cdba \end{array}
\qquad
d\left\{\begin{array}{l} da\left\{\begin{array}{l} dab \\ dac \end{array}\right. \\ db\left\{\begin{array}{l} dba \\ dbc \end{array}\right. \\ dc\left\{\begin{array}{l} dca \\ dcb \end{array}\right. \end{array}\right. \quad \begin{array}{l} dabc \\ dacb \\ dbac \\ dbca \\ dcab \\ dcba \end{array}
$$

The permutations two at a time are formed from those one at a time, by annexing to each of the latter each remaining letter in turn; those three at a time from those two at a time in like manner; and so on. Evidently the permutations thus formed are all different.

Of four things, only four permutations one at a time can be formed. And since, in the permutations two at a time formed from those one at a time, each thing is followed by each remaining thing, none of those two at a time are omitted. For a similar reason, none of those three and four at a time are omitted. Therefore the above representation includes all permutations of the four letters, one, two, three, and four at a time.

5. The number of permutations of n things taken r at a time is denoted by the symbol ${}_nP_r$.

Then from the enumeration of the preceding article, we have

$$ {}_4P_1 = 4,\ {}_4P_2 = 12,\ {}_4P_3 = 24,\ {}_4P_4 = 24. $$

6. When the number of things is large, the preceding method of enumeration becomes laborious.

The following example illustrates a method of deriving a general formula for $_nP_r$.

We have $$_4P_1 = 4.$$

Each permutation one at a time gives as many permutations two at a time as there are things remaining to annex to it in turn, in this case three.

Therefore $$_4P_2 = {_4P_1} \times 3 = 4 \times 3.$$

Each permutation two at a time gives as many permutations three at a time as there are things remaining to annex to it in turn, in this case two.

Therefore $$_4P_3 = {_4P_2} \times 2 = 4 \times 3 \times 2.$$

In like manner, $_4P_4 = {_4P_3} = 4 \times 3 \times 2 \times 1.$

In general, $_nP_r = n(n-1)(n-2)\cdots(n-r+1)$,

when the n things are all different.

Evidently $$_nP_1 = n. \tag{1}$$

From each permutation of n things one at a time we obtain, by annexing to it each of the $n-1$ remaining things in turn, $n-1$ permutations two at a time.

Therefore $$_nP_2 = {_nP_1}(n-1) = n(n-1). \tag{2}$$

Again, from each permutation of n things two at a time we obtain, by annexing to it each of the $n-2$ remaining things in turn, $n-2$ permutations three at a time.

Therefore $$_nP_3 = {_nP_2}(n-2) = n(n-1)(n-2). \tag{3}$$

In like manner,

$$_nP_4 = {_nP_3}(n-3) = n(n-1)(n-2)(n-3). \tag{4}$$

The method is evidently general. The number subtracted from n in the last factor in (1)–(4) is *one less than the number of things taken at a time.* Therefore,

$$_nP_r = n(n-1)(n-2)\cdots[n-(r-1)] = n(n-1)(n-2)\cdots(n-r+1).$$

7. Observe that the number of factors in the formula for ${}_nP_r$ is equal to the number of things taken at a time.

E.g., $${}_8P_5 = 8 \times 7 \times 6 \times 5 \times 4 = 6720.$$

8. If all the things are taken at a time, *i.e.*, if $r = n$, we have

$${}_nP_n = n(n-1)(n-2)\cdots(n-n+1) = n(n-1)(n-2)\cdots 3 \times 2 \times 1.$$

E.g., $${}_5P_5 = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

9. The continued product

$$n(n-1)(n-2)\cdots 3 \times 2 \times 1$$

is called **Factorial-n**, and is denoted by the symbol $\lfloor n$ or $n!$

Therefore the formula of the preceding article may be written

$${}_nP_n = \lfloor n.$$

E.g., ${}_7P_7 = \lfloor 7$, or $7!$, $= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1.$

10. In many applications the things considered are not all different. We will now derive a formula for the number of permutations of n things, *taken all at a time*, when some of them are alike.

Let p of the n things be alike, and suppose the permutations n at a time to be formed. In any one of these permutations, let the p like things be replaced by p unlike things, different from all the rest. Then by changing the order of these p new things only, we can form $\lfloor p$ permutations from the one permutation. In like manner, $\lfloor p$ permutations can be formed from each of the given permutations. Therefore

$${}_nP_n \text{ (all different)} = {}_nP_n \times \lfloor p \, (p \text{ alike}),$$

or $${}_nP_n\,(p \text{ alike}) = \frac{{}_nP_n}{\lfloor p} = \frac{\lfloor n}{\lfloor p}.$$

In like manner, it can be proved that

$${}_nP_n(p \text{ alike, } q \text{ alike, } \cdots) = \frac{\lfloor n}{\lfloor p \times \lfloor q \times \cdots}.$$

E.g., $${}_8P_8(3 \text{ alike}) = \frac{\lfloor 8}{\lfloor 3} = 6720.$$

EXERCISES I.

Find the values of

1. ${}_{18}P_4$. 2. ${}_{15}P_3$. 3. ${}_{10}P_{10}$. 4. ${}_{20}P_5$.

5. ${}_{n+1}P_3$. 6. ${}_{2n+1}P_5$. 7. ${}_{n+1}P_{n-1}$. 8. ${}_{n+k}P_k$.

Find the value of n, when

9. ${}_nP_4 = 3\,{}_nP_3$. 10. ${}_nP_6 = 20\,{}_nP_4$. 11. ${}_{n+2}P_4 = 15\,{}_nP_3$.

12. ${}_{n+1}P_4 = 30\,{}_{n-1}P_2$. 13. ${}_{n+4}P_3 = 8\,{}_{n+3}P_2$. 14. ${}_{2n+1}P_4 = 140\,{}_nP_3$.

Find the value of k, when

15. ${}_{10}P_{k+6} = 3\,{}_{10}P_{k+5}$. 16. ${}_7P_{k+1} = 12\,{}_7P_{k-1}$. 17. ${}_{12}P_k = 20\,{}_{12}P_{k-2}$.

18. How many numbers of 4 figures can be formed with 1, 2, 3, 4, 5, 6, 7?

19. How many numbers of 4 figures can be formed with 0, 1, 2, 3, 4, 5, 6, 7?

20. How many even numbers of 4 figures can be formed with 4, 5, 3, 2?

21. In how many ways can 6 pupils be seated in 10 seats?

22. How many numbers of 5 figures can be formed with 1, 2, 3, 4, 5, 6, 7, 8, 9, if the figure 7 be in the middle of each number?

23. How many permutations can be formed with the letters in the word *Philippine?*

24. How many permutations can be formed with the letters in the word *Iloilo?*

25. In how many ways can 7 men be seated at a round table?

26. In how many ways can a bracelet be made by stringing together 7 pearls of different shades?

COMBINATIONS.

11. The formula for the number of combinations of n things, r at a time, which is denoted by ${}_nC_r$, is most readily obtained by deriving a relation between ${}_nP_r$ and ${}_nC_r$. The method will be illustrated by a particular example.

The combinations of the four letters a, b, c, d, taken three at a time, evidently are: abc, abd, acd, bcd. From the combination abc we obtain, by changing the order of the letters in all possible ways, $\lfloor 3$ permutations. In like manner, each of the combinations gives $\lfloor 3$ permutations.

Therefore

$$_4P_3 = {}_4C_3 \times \lfloor 3, \text{ or } {}_4C_3 = \frac{_4P_3}{\lfloor 3} = \frac{4 \times 3 \times 2}{\lfloor 3}.$$

In general,

$$_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r},$$

wherein the n *things are all different.*

For, from each combination that contains r things can be formed $\lfloor r$ permutations, by changing the order of the things in all possible ways. Therefore

$$_nP_r = {}_nC_r \times \lfloor r, \text{ or } {}_nC_r = \frac{_nP_r}{\lfloor r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}.$$

E.g.,
$$_8C_3 = \frac{8 \times 7 \times 6}{\lfloor 3} = 56.$$

12. The formulæ for $_nC_1$, $_nC_2$, $_nC_3$, $\cdots$, $_nC_r$ may be represented by the following abbreviations:

$$n = \frac{n}{1} = \binom{n}{1}, \ \frac{n(n-1)}{1 \cdot 2} = \binom{n}{2}, \ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = \binom{n}{3},$$
$$\cdots, \ \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} = \binom{n}{r}.$$

Observe that in the symbolic notation the upper number is the number of things, and the lower number is the number taken at a time.

E.g.,
$$_7C_4 = \binom{7}{4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = 35.$$

13. The formula for $_nC_r$ can be put in a more convenient form for purposes of theory.

We have

$$_nC_r = \frac{n(n-1)\cdots(n-r+1)\times(n-r)(n-r-1)\cdots 3\times 2\times 1}{\underline{|r}\times(n-r)(n-r-1)\cdots 3\times 2\times 1}$$

$$=\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

14. We have $$_nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}},$$

and $$_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}.$$

Therefore, $$_nC_r = {_nC_{n-r}}.$$

That is, *the number of combinations of n dissimilar things r at a time is equal to the number of combinations of the n things $n-r$ at a time.*

This relation is also evident from the definition of a combination. For, every time that r things are taken from the n things to form a combination, there is left a combination of $n-r$ things.

E.g., $$_{100}C_{98} = {_{100}C_2} = \frac{100\times 99}{1\times 2} = 4950.$$

This relation is thus useful in computing the number of combinations when the number of things taken at a time is large.

15. The Greatest Value of $_nC_r$. — We have

$$_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}\times\frac{n-r+1}{r}$$

$$= {_nC_{r-1}}\times\frac{n-r+1}{r} = {_nC_{r-1}}\left(\frac{n+1}{r}-1\right). \qquad (1)$$

Also, $$_nC_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)}{1\cdot 2\cdot 3\cdots r(r+1)}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}\times\frac{n-r}{r+1}$$

$$= {_nC_r}\times\frac{n-r}{r+1}.$$

Whence $$ {}_nC_r = {}_nC_{r+1} \times \frac{r+1}{n-r}. \tag{2}$$

From (1),

$$ {}_nC_r > {}_nC_{r-1}, \text{ when } \frac{n+1}{r} - 1 > 1, \text{ or } r < \frac{n+1}{2}. \tag{3}$$

That is, the number of combinations of n things, taken any number less than $\frac{n+1}{2}$ at a time, is greater than the number of combinations taken one less at a time, and therefore greater than the number of combinations taken any number less at a time.

From (2), $$ {}_nC_r > {}_nC_{r+1}, \text{ when } \frac{r+1}{n-r} > 1, \text{ or } r > \frac{n-1}{2}. \tag{4}$$

That is, the number of combinations of n things, taken any number greater than $\frac{n-1}{2}$ at a time, is greater than the number of combinations taken one more at a time, and therefore greater than the number of combinations taken any number more at a time. Consequently, ${}_nC_r$ is greatest when r, an integer, lies between $\frac{n-1}{2}$ and $\frac{n+1}{2}$.

(i) n *Even.* Let $n = 2m$. Then r is an integer in value between $\frac{2m-1}{2}, = m - \frac{1}{2}$, and $\frac{2m+1}{2}, = m + \frac{1}{2}$. That is, $r = m, = \frac{n}{2}$. Therefore, when n is even, the greatest number of combinations is ${}_nC_{\frac{n}{2}}$.

(ii) n *Odd.* Let $n = 2m+1$. Then r should have an integral value between $\frac{2m}{2}, = m$, and $\frac{2m+2}{2}, = m+1$. This is evidently impossible, since m and $m+1$ are consecutive integers.

But, when $r = m, < m+1, = \frac{n+1}{2}$, then by (3),

$$ {}_{2m+1}C_m > {}_{2m+1}C_{m-1}; $$

and, when $r = m+1, > m, = \frac{n-1}{2}$, then by (4),

$$ {}_{2m+1}C_{m+1} > {}_{2m+1}C_{m+2}. $$

Also, by Art. 14, $\quad {}_{2m+1}C_m = {}_{2m+1}C_{(2m+1-m)} = {}_{2m+1}C_{m+1}.$

Consequently, when n is odd, the greatest number of combinations is

$${}_{2m+1}C_{m}, = {}_{2m+1}C_{m+1}, \text{ or } {}_nC_{\frac{n-1}{2}}, = {}_nC_{\frac{n+1}{2}}.$$

Ex. 1. When $n = 4$, the greatest number of combinations is ${}_4C_2$.

We have $\quad {}_4C_1 = 4,\ {}_4C_2 = 6,\ {}_4C_3 = 4,\ {}_4C_1 = 1.$

Ex. 2. When $n = 5$, the greatest number of combinations is

$${}_5C_2, = {}_5C_3.$$

We have $\quad {}_5C_1 = 5,\ {}_5C_2 = 10,\ {}_5C_3 = 10,\ {}_5C_4 = 5,\ {}_5C_5 = 1$

EXERCISES II.

Find the values of

1. ${}_{11}C_5$. 2. ${}_{15}C_7$. 3. ${}_{25}C_{20}$. 4. ${}_{98}C_{95}$. 5. ${}_nC_{n-5}$.

Find the value of n, when

6. $2\,{}_nC_5 = 9\,{}_{n-2}C_5$. 7. $3\,{}_nC_3 = 10\,{}_{n-2}C_2$. 8. $4\,{}_{n+1}C_4 = 15\,{}_{n-1}C_3$.

9. ${}_{n+1}P_4 = 112\,{}_{n-1}C_2$. 10. ${}_{n+1}P_4 = 84\,{}_{n-1}C_3$. 11. ${}_nP_3 = 24\,{}_nC_{n-1}$.

Find the value of k, when

12. ${}_8P_k = 24\,{}_8C_k$. 13. ${}_6P_{k+1} = 48\,{}_6C_k$. 14. ${}_{10}P_k = 144\,{}_{10}C_{k-1}$.

15. Prove that ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

16. In how many ways can a committee of 4 men be appointed from 25 men?

17. In how many ways can 3 books be selected from 15 books?

18. In a plane are 20 points, no 3 of which are in the same straight line. How many triangles can be formed with 3 points as vertices? How many quadrilaterals, with 4 points as vertices? How many hexagons, with 6 points as vertices?

Find the values of r and ${}_nC_r$, when ${}_nC_r$ is greatest, in

19. ${}_7C_r$. 20. ${}_8C_r$. 21. ${}_{11}C_r$. 22. ${}_{14}C_r$. 23. ${}_{17}C_r$.

TWO IMPORTANT PRINCIPLES.

16. The following example illustrates an important principle.

Pr. Between two cities A and B there are five railroad lines. In how many ways can a man go from A to B and return by a different road?

He can go to B in either of five ways. With each of these five ways he has a choice of four ways of returning. Hence he can make the round trip in $5 \times 4, = 20$, ways.

Evidently, if he were not required to return by a different road he could make the trip in $5 \times 5, = 25$, ways.

The general principle is:

*If one thing can be done in **a** ways, and another thing can be done in **b** ways, and the doing of the first thing does not interfere with the doing of the second, the two things can be done in **ab** ways.*

The truth of the principle is evident.

17. The following relation will be useful in subsequent work:

$$ {}_{m+n}C_r = {}_mC_r + {}_mC_{r-1}\,{}_nC_1 + {}_mC_{r-2}\,{}_nC_2 + \cdots + {}_mC_2\,{}_nC_{r-2} + {}_mC_1\,{}_nC_{r-1} + {}_nC_r, \qquad (1) $$

in which $m >$ or $= r$, $n >$ or $= r$.

The number of combinations of the $m + n$ things r at a time is evidently the sum of:

The number of combinations of m things taken r at a time, or ${}_mC_r$.

The number of combinations of m things taken $r-1$ at a time, multiplied by the number of combinations of n things taken one at a time, or ${}_mC_{r-1}\,{}_nC_1$. And so on.

This relation may be written

$$ \binom{m+n}{r} = \binom{m}{r} + \binom{m}{r-1}\binom{n}{1} + \cdots + \binom{m}{1}\binom{n}{r-1} + \binom{n}{r}. \qquad (2) $$

18. The relation of the preceding article requires m, n, and r to be integers. Evidently, however, the second member of

(2) could be made identical with the first member by ordinary reduction. We, therefore, conclude that this relation holds for all rational values of m and n, provided r is a positive integer.

PROBLEMS.

19. Pr. **1.** In how many ways can a committee of 3 Republicans and 4 Democrats be appointed from 18 Republicans and 12 Democrats?

The 3 Republicans can be chosen in ${}_{18}C_3, = 816$, ways, and the 4 Democrats in ${}_{12}C_4, = 495$, ways. Since any 3 Republicans can be associated with any 4 Democrats to form the committee, the required number of ways is $816 \times 495, = 403{,}920$.

Pr. **2.** A box contains 20 balls numbered 1 to 20. In how many ways can 7 balls be selected, if 1 be included, and 2, 3 be excluded?

We first set aside 1 to be included, and 2, 3 to be excluded, and from the remaining 17 balls select 6 balls. Then 1 may be combined with each of the latter in one way, giving a combination of 7 balls. Therefore the problem is equivalent to determining the number of combinations of 17 things, 6 at a time.

Hence $${}_{17}C_6 = \frac{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, = 12376,$$

is the required number of ways.

EXERCISES III.

1. A man has 3 coats, 4 vests, and 5 pairs of trousers. In how many ways can he dress?

2. In how many ways can 4 white balls, 3 black balls, and 2 red balls be selected from 8 white balls, 7 black balls, and 5 red balls?

3. In how many ways can permutations be formed, with 10 consonants and 4 vowels, each one to contain 5 consonants and 2 vowels?

4. In how many ways can 4 hearts, 3 diamonds, 2 clubs, and 1 spade be drawn from a pack containing 13 cards of each kind?

5. How many numbers of 7 figures can be formed with 1, 2, 3, 4, 5, 6, 7, if the figures 4, 5, 6 be kept together?

6. How many permutations of 10 letters can be formed from 5 consonants and 5 vowels, if no two consonants be adjacent?

7. How many permutations of 9 letters can be formed from 5 consonants and 4 vowels, if each vowel be placed between two consonants?

8. In a school are 96 pupils. In how many ways can a teacher divide them into sections of 12?

9. In how many ways can 4 ladies and 4 gentlemen be seated at a square table, so that a gentleman and a lady shall be seated at each side?

10. How many throws can be made with 2 dice, if such throws as 1, 2 and 2, 1 be regarded as the same? How many with 3 dice?

11. In how many ways can the sum 10 be thrown with 2 dice? With 3 dice?

12. A box contains 15 balls, numbered 1 to 15. In how many ways can 5 balls be selected, if 1, 2, 3 be included? In how many ways, if 1, 2 be included, and 3 excluded? In how many ways, if any two of the numbers 1, 2, 3 be included, the other excluded?

13. In how many ways can 10 different coins be arranged in a row, if the faces of the coins are distinct? In how many ways can they be arranged in a circle?

14. In how many ways can a number of 6 figures be formed with 1, 1, 1, 2, 2, 3, the first and last figure of each number to be an even digit?

15. In how many ways can 7 gentlemen and 10 ladies arrange a game of lawn tennis, each side to consist of 1 lady and 1 gentleman?

CHAPTER XXIV.

VARIABLES AND LIMITS.

VARIABLES.

1. A **Variable** is a number that may have a series of different values in the same investigation or problem.

A **Constant** is a number that has a fixed value in an investigation or problem.

Thus, if d be the number of feet a body has fallen from rest in s seconds, it has been shown by experiment that

$$d = 16\,s^2.$$

As the body falls, the distance d and the time s are variables, and 16 is a constant.

Again, time measured from a past date is a variable, while time measured between two fixed dates is a constant.

2. The constants in a mathematical investigation are, as a rule, general numbers, and are represented by the first letters of the alphabet, a, b, c, etc.; variables are usually represented by the last letters, x, y, z, etc.

3. A variable which has a definite value, or set of values, corresponding to a value of a second variable, is called a **Function** of the latter.

Thus, $16\,x^2$, $\pm\sqrt{(a^2 - x^2)}$, etc., are functions of x; corresponding to any value of x, the first function has one value, the second has two values.

Again, the area of a circle is a function of its radius; the distance a train runs is a function of the time and speed.

4. Much simplicity is introduced into mathematical investigations by employing special symbols for functions.

The symbol $f(x)$, read *function of* x, is very commonly used to denote a function of x.

Thus, $f(x)$ may denote x^2+1 in one investigation, ax^2+bx+c in another.

5. The result of substituting a particular value for the variable in a given expression may be indicated by substituting the same value for the variable in the functional symbol.

Thus, if $f(x)=x^2+1$, then $f(a)=a^2+1, f(2)=2^2+1=5$, $f(0)=0+1=1$.

EXERCISES I.

1. Given $f(x)=5x^2-3x+2$; find $f(3), f(0), f(-4), f(x^2)$.
2. Given $f(x)=(x-a)(x-b)(x-c)$; find $f(a), f(b)$.
3. Given $f(x)=x^2+1$; find $f(x^2)$, $[f(x)]^2$.
4. Given $f(x)=x^2-3x+2$; find $f(x+4)$, $f(x+h)$.
5. Given $f(x)=a^x$; find $f(0)$, $f(4)$, $f(-5)$, $f(x^2)$, $f(a)$.
6. Given $f(x)=x^3+px^2+qx+r$; find $f\left(y-\frac{p}{3}\right)$.
7. Given $f(m)=1+mx+\frac{m(m-1)}{\underline{|2}}x^2+\cdots$;

   find $f(5)$, $f(\frac{3}{5})$, $f(-3)$, $f(0)$.

LIMITS.

6. When the difference between a variable and a constant may become and remain less than any assigned positive number, however small, the constant is called the **Limit** of the variable.

Let the point P move from A towards B (Fig. 1) in the following way: First to P_1, one-half of the distance from A to B; next from P_1 to P_2, one-half of the distance from P_1 to B;

then from P_2 to P_3, one-half of the distance from P_2 to B; and so on.

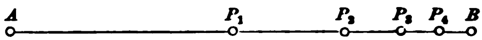

FIG. 1.

Evidently, as P thus moves from A to B, its variable distance from A becomes more and more nearly equal to AB, and the difference between AP and AB can be made less than any assigned distance, however small, by continuing indefinitely the motion of P. Therefore AB is the limit of the variable AP.

If we call the distance from A to B unity, we have

$$AP_1 = \tfrac{1}{2},\ P_1P_2 = \tfrac{1}{4},\ P_2P_3 = \tfrac{1}{8},\ P_3P_4 = \tfrac{1}{16},\ \cdots.$$

Hence,

$$AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + \cdots = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots.$$

But, by Ch. XXI, Art. 25, the variable sum of the series on the right approaches 1 as a limit. That is,

$$\text{limit of } (AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + \cdots) = AB.$$

7. It follows from the definition of a limit that the variable may be always greater, or always less, or sometimes greater and sometimes less than its limit.

Thus, by Ch. XXI, Art. 25, we have

$$\text{limit } (1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{8} - \cdots) = 0, \qquad (1)$$

$$\text{limit } (1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots) = 2, \qquad (2)$$

$$\text{limit } (1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots) = \tfrac{2}{3}. \qquad (3)$$

And in (1), $S_1 = 1,\ S_2 = \tfrac{1}{2},\ S_3 = \tfrac{1}{4},\ S_4 = \tfrac{1}{8},\ \cdots;$ (4)

in (2), $S_1 = 1,\ S_2 = \tfrac{3}{2},\ S_3 = \tfrac{7}{4},\ S_4 = \tfrac{15}{8},\ \cdots;$ (5)

in (3), $S_1 = 1,\ S_2 = \tfrac{1}{2},\ S_3 = \tfrac{3}{4},\ S_4 = \tfrac{5}{8},\ \cdots.$ (6)

Evidently the variable in each of these examples is the sum, which changes as the number of terms increases.

8. The symbol, $\doteq$, read *approaches as a limit,* or simply *approaches,* is placed between a variable and its limit.

The word limit may be abbreviated to lim.

Thus, $\lim_{x \doteq 1}(1-x)=0$, read *the limit of* $1-x$, *as* x *approaches* 1, *is* 0.

Infinites and Infinitesimals.

9. The following considerations lead to important mathematical concepts:

The fractions

$$\frac{2}{.1}, = 20;\ \frac{2}{.01}, = 200;\ \frac{2}{.001}, = 2000;\ \frac{2}{.0001}, = 20000;\ \text{etc.},$$

are particular values of the fraction $\frac{n}{x}$, in which the denominator x is assumed to be a variable. It is evident that the value of this fraction can be made greater than any assigned number, however great, by taking its denominator sufficiently small.

A variable which can become and remain numerically greater than any assigned positive number, however great, is called an **Infinite Number,** or simply an **Infinite.**

An infinite variable is denoted by the symbol ∞.

10. The numbers, variables and constants, which have been hitherto used in this book are, for the sake of distinction, called **Finite Numbers.**

11. The fractions

$$\frac{2}{10}, = .2;\ \frac{2}{100}, = .02;\ \frac{2}{1000}, = .002;\ \frac{2}{10000}, = .0002;\ \text{etc.}$$

are also particular values of the fraction $\frac{n}{x}$, in which, as above, the denominator x is assumed to be a variable. It is evident that the value of the fraction $\frac{n}{x}$ can also be made less than any assigned number, however small, by taking the denominator sufficiently great.

A variable which can become and remain numerically less than any assigned positive number, however small, is called an **Infinitesimal.**

No symbol by which to denote an infinitesimal variable has been generally adopted.

It follows from the definition that the limit of an infinitesimal is 0.

12. It is important to keep in mind that both infinites and infinitesimals are *variables*. Their definitions imply that *fixed* values cannot be assigned to them.

An infinitesimal should therefore not be confused with 0, which is the *constant* difference between any two equal numbers.

13. The statement, *x becomes infinite*, or *x increases numerically beyond any assigned positive number, however great,* is frequently abbreviated by the expression, $x \doteq \infty$.

14. The conclusions reached in Arts. 9 and 11 can now be restated thus:

(i.) *If the numerator of a fraction remain finite and not* 0, *and the denominator approach zero, the value of the fraction will become infinite;* or stated symbolically,

$$\frac{n}{x} \doteq \infty, \text{ as } x \doteq 0,$$

wherein n is finite and not 0.

(ii.) *If the numerator of a fraction remain finite and not* 0, *and the denominator become infinite, the value of the fraction will approach* 0; or stated symbolically,

$$\frac{n}{x} \doteq 0, \text{ as } x \doteq \infty,$$

wherein n is finite and not 0.

Observe that these principles hold not only when n is a constant, not 0, but also when n is a variable, provided it does not become infinite.

15. *The difference between a variable and its limit is evidently an infinitesimal;* that is,

$$\text{if } \lim x = a, \text{ then } \lim (x - a) = 0.$$

Consequently, if $\lim x = a$, we have

$$x - a = x', \text{ or } x = a + x',$$

wherein x' is a variable whose limit is 0.

Conversely, if $x = a + x'$, and x' be a variable whose limit is 0, then $\lim x = a$.

16. *If the limit of a variable be* 0, *the limit of the product of the variable and any finite number is* 0; that is,

if $\lim x = 0$, and a be any finite number, $\lim ax = 0$.

Let k be any number, however small. Then x can be made less numerically than $\frac{k}{a}$, and, therefore, ax less than k. Hence, $\lim ax = 0$.

Fundamental Principles of Limits.

17. (i.) *If two variables be always equal, and one of them approach a limit, the other approaches the same limit.* That is,

if $x = y$, and $x \doteq a$, then $y \doteq a$.

(ii.) *If two variables be always equal as they approach their limits, their limits are equal.* That is,

if $\lim x = a$, $\lim y = b$, and $x = y$, then $a = b$.

(iii.) *The limit of the algebraical sum of a finite number of variables is the sum of their limits.* That is,

if $\lim x = a$, $\lim y = b$, $\cdots$, then $\lim (x + y + \cdots) = a + b + \cdots$.

(iv.) *The limit of the product of a finite number of variables is the product of their limits, if none of the limits be* ∞. That is,

if $\lim x = a$, $\lim y = b$, $\cdots$, then $\lim (xy \cdots) = ab \cdots$.

(v.) *The limit of the quotient of two variables is the quotient of their limits, if the limit of the divisor be not* 0. That is,

if $\lim x = a$, $\lim y = b$, then $\lim \left(\frac{x}{y}\right) = \frac{a}{b}$, when $\lim y \neq 0$.

The proofs follow:

(i.) We have $x = a + x'$, wherein, by Art. 15, x' is a variable whose limit is 0. Then, since $y = x$ always, we have $y = a + x'$. Hence $\lim y = a$.

(ii.) This principle follows directly from (i.).

(iii.) We have $x = a + x', y = b + y', \cdots$, wherein, by Art. 15, $x', y', \cdots$ are variables whose limits are 0.

Then $x + y + \cdots = (a + b + \cdots) + (x' + y' + \cdots)$.

Let k be any assigned number, however small. Then each of the variables $x', y', \cdots$ can be made less than $\frac{k}{n}$, wherein n is the number of variables. Therefore, $x' + y' + \cdots$ can be made less than k. Consequently $\lim (x + y \cdots) = a + b + \cdots$.

(iv.) We have $x = a + x'$, $y = b + y', \cdots$, wherein $x', y', \cdots$ are variables whose limits are 0, and $a, b, \cdots$ are finite.

Then $xy = ab + bx' + ay' + x'y'$. Therefore, by (iii.),

$$\lim xy + = \lim ab + \lim bx' + \lim ay' + \lim x'y'$$
$$= ab, \text{ since } \lim bx' = 0, \cdots, \text{ by Art. 16.}$$

In like manner, the principle can be proved for any finite number of factors.

(v.) Let $\frac{x}{y} = q$, or $x = yq$. Then, by (iv.), $\lim x = \lim y \lim q$.

Therefore, $\quad \lim q = \frac{\lim x}{\lim y}$, or $\lim \frac{x}{y} = \frac{\lim x}{\lim y}$.

Indeterminate Fractions.

18. It follows from the definition of a fraction that $\frac{0}{0}$ is a number which multiplied by 0 gives 0. But any finite number multiplied by 0 gives 0, or $0n = 0$. Consequently $\frac{0}{0}$ may denote *any number whatever*.

For this reason, such a fraction is called an **Indeterminate Fraction.**

19. The fraction $\frac{x^2 - 9}{x - 3}$ becomes $\frac{0}{0}$ when $x = 3$, and has no definite value. But as long as $x \neq 3$, however little it may differ from 3, we may perform the indicated division. We therefore have

$$\frac{x^2 - 9}{x - 3} = x + 3, \text{ when } x \neq 3.$$

Now since the limit of the fraction depends upon values of x which differ from 3, however little, we have

$$\lim_{x \doteq 3} \frac{x^2-9}{x-3} = \lim_{x \doteq 3} (x+3) = 6.$$

Although the given fraction is indeterminate, it is clearly desirable that it shall have a definite value. *We therefore assign to* $\frac{x^2-9}{x-3}$ *the value* 6, *when* $x = 3$.

That is, we *define* an indeterminate fraction to be the limit of the fraction as the variable approaches that value which renders it indeterminate. In this way we may obtain a definite value when the fraction involves but one variable.

20. The fraction $\frac{\infty}{\infty}$ is a number which multiplied by ∞ gives ∞. But any finite number multiplied by ∞ gives ∞. Therefore $\frac{\infty}{\infty}$ is also an *indeterminate fraction.*

21. The fraction $\frac{n-1}{n+1}$ becomes $\frac{\infty}{\infty}$, as $n \doteq \infty$. Dividing numerator and denominator by n, we have

$$\frac{n-1}{n+1} = \frac{1-\frac{1}{n}}{1+\frac{1}{n}}.$$

Since $$\frac{1}{n} \doteq 0, \text{ as } n \doteq \infty,$$

$$\lim_{n \doteq \infty} \frac{n-1}{n+1} = \lim_{n \doteq \infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}} = 1.$$

EXERCISES II.

Find the limiting values of the following fractions:

1. $\frac{x^2-6x+5}{x^2-8x+15}$, when $x \doteq 5$.

2. $\frac{x^2-3x+2}{x^2+x-6}$, when $x \doteq 2$.

3. $\frac{3a^2-ab-2b^2}{9a^2+12ab+4b^2}$, when $a \doteq -\frac{2}{3}b$.

4. $\dfrac{9x^2 - 30xy + 25y^2}{3x^2 - 2xy - 5y^2}$, when $x \doteq \frac{5}{3}y$.

5. $\dfrac{x^3 + 2x^2 - x - 2}{x^2 + x - 2}$, when $x \doteq 1$.

6. $\dfrac{(x^2 - 1)^2}{x - 1}$, when $x \doteq 1$.

7. $\dfrac{x^3 - 3x + 2}{x^2 - 7x + 5}$, when $x \doteq 1$.

8. $\dfrac{x^2 - 4x + 5}{x^3 - 3x + 2}$, when $x \doteq 1$.

9. $\dfrac{a^{2x} - 1}{a^x - 1}$, when $x \doteq 0$.

10. $\dfrac{2x + 1}{(4x^2 - 1)^3}$, when $x \doteq -\frac{1}{2}$.

Find the limiting values of the following fractions when $n \doteq \infty$:

11. $\dfrac{n + 1}{n^2 - 1}$.

12. $\dfrac{n^2 - 9}{n + 3}$.

13. $\dfrac{n(n - 1)}{\lfloor 2} \cdot \dfrac{1}{n^2}$.

14. $\dfrac{n^2 - 3n + 2}{2n^2 - 3n + 4}$.

15. $\dfrac{nx(nx - 1)(nx - 2)}{\lfloor 3} \cdot \dfrac{1}{n^3}$.

Indeterminate Solutions.

22. The preceding principles may be further illustrated by examining the infinite and indeterminate solutions of certain problems.

Pr. A merchant buys four pieces of goods. In the second piece there are 3 yards less than in the first, in the third 7 yards less than in the first, and in the fourth 10 yards less than in the first. The number of yards in the first and fourth is equal to the number of yards in the second and third. How many yards are there in the first piece?

Let x stand for the number of yards in the first piece; then the number of yards in the second piece is $x - 3$; in the third piece, $x - 7$; in the fourth piece, $x - 10$. Therefore, by the condition of the problem, we have

$$x + (x - 10) = (x - 3) + (x - 7), \text{ or } 2x - 10 = 2x - 10.$$

This equation is an identity, and is therefore satisfied by *any finite value of x.*

If it be solved in the usual way, we obtain

$$(2-2)x = 10 - 10, \text{ or } x = \frac{10-10}{2-2} = \frac{0}{0}.$$

That is, the conditions of the problem will be satisfied by any number of yards in the first piece.

Infinite Solutions.

23. Pr. A cistern has three pipes. Through the first it can be filled in 24 minutes; through the second in 36 minutes; through the third it can be emptied in a minutes. In what time will the cistern be filled if all the pipes be opened at the same time?

Let x stand for the number of minutes after which the cistern will be filled. In one minute $\frac{1}{24}$ of its capacity enters through the first pipe, and hence in x minutes $\frac{1}{24}x$ of its capacity enters. For a similar reason, $\frac{1}{36}x$ of its capacity enters through the second pipe in x minutes; and in the same time $\frac{1}{a}x$ of its capacity is discharged through the third pipe.

Therefore, after x minutes there is in the cistern

$$\frac{x}{24} + \frac{x}{36} - \frac{x}{a}, = (\tfrac{5}{72} - a)x,$$

of its capacity. But by the condition of the problem, that the cistern is then filled, we have

$$(\tfrac{5}{72} - a)x = 1;$$

whence $$x = \frac{1}{\frac{5}{72} - a}.$$

If we now let a approach $\frac{5}{72}$, then x becomes infinite.

This result would mean that the cistern would never be filled. This is also evident from the data of the problem, since the third pipe in a given time would discharge from the cistern as much as would enter it through the other pipes.

The Problem of the Couriers.

24. Pr. Two couriers are travelling along a road in the direction from M to N; one courier at the rate of m_1 miles an hour, the other at the rate of m_2 miles an hour. The former

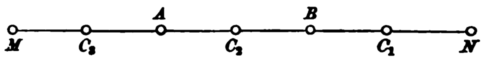

FIG. 2.

is seen at the station A at noon, and the other is seen h hours later at the station B, which is d miles from A in the direction in which the couriers are travelling. Where do the couriers meet?

Assume that they meet to the right of B at a point C_1, and let x stand for the number of miles from B to the place of meeting C_1 (Fig. 2).

The first courier, moving at the rate of m_1 miles an hour, travels $d + x$ miles, from A to C_1, in $\dfrac{d+x}{m_1}$ hours; the second courier, moving at the rate of m_2 miles an hour, travels x miles, from B to C_1, in $\dfrac{x}{m_2}$ hours. By the condition of the problem it is evident that, if the place of meeting is to the right of B, the number of hours it takes the first courier to travel from A to C_1 exceeds by h the number of hours it takes the second courier to travel from B to C_1. We therefore have

$$\frac{d+x}{m_1} - \frac{x}{m_2} = h,$$

whence
$$x = \frac{hm_1m_2 - dm_2}{m_2 - m_1} = \frac{m_2(hm_1 - d)}{m_2 - m_1}.$$

(i.) **A Positive Result.** — The result will be positive either when $hm_1 > d$ and $m_2 > m_1$, or when $hm_1 < d$ and $m_2 < m_1$. A positive result means that the problem is possible with the assumption made; *i.e.*, that the couriers meet at a point to the right of B.

(ii.) **A Negative Result.** — The result will be negative either when $hm_1 > d$ and $m_2 < m_1$, or when $hm_1 < d$ and $m_2 > m_1$. Such

a result shows that the assumption that the couriers meet to the right of B is untenable, since, as we have seen, in that case the result is positive.

That under the assumed conditions the couriers can meet only at some point to the left of B can also be inferred from the following considerations, which are independent of the negative result: If $hm_1 > d$, the first courier has passed B when the second courier is seen at that station; that is, the second courier is behind the first at that time. And since also $m_2 < m_1$, the first courier is travelling the faster, and must therefore have overtaken the second, and at some point to the left of B.

On the other hand, if $hm_1 > d$, the first courier has not yet reached B when the second is seen at that station; that is, the first courier is behind the second at that time. And since also $m_2 > m_1$, the second courier is travelling the faster, and must therefore have overtaken the first, at some point to the left of B. Similar reasoning could have been applied in (i.).

(iii.) **A Zero Result.** — A zero result is obtained when $hm_1 = d$, and m_2 is not equal to m_1; that is, the meeting takes place at B. This is also evident from the assumed conditions. For the first courier reaches B h hours after he was seen at A; and since the second courier is seen at B h hours after the first was seen at A, the meeting must take place at B.

(iv.) **Indeterminate Result.** — An indeterminate result is obtained if $hm_1 \doteq d$, and $m_2 \doteq m_1$. In this case every point of the road can be regarded as their place of meeting. For the first courier evidently reaches B at the time at which the second courier is seen at that station; and since they are travelling at the same rate, they must be together all the time. The problem under these conditions becomes indeterminate.

(v.) **An Infinite Result.** — An infinite result is obtained when $hm_1 \neq d$, and $m_2 \doteq m_1$. In this case a meeting of the couriers is impossible, since both travel at the same rate, and when the second is seen at B the first either has not yet reached B or has already passed that station.

An infinite result also means that the more nearly equal m_1 and m_2 are, the further removed is the place of meeting.

EXERCISES III.

Solve the following problems, and interpret the results:

1. In a number of two digits, the digit in the tens' place exceeds the digit in the units' place by 5. If the digits be interchanged, the resulting number will be less than the original number by 45. What is the number?

2. The sum of the first and third of three consecutive even numbers is equal to twice the second. What are the numbers?

3. A father is 26 years older than his son, and the sum of their ages is 26 years less than twice the father's age. How old is the son?

4. In a number of two digits, the digit in the units' place exceeds the digit in the tens' place by 4. If the sum of the digits be divided by 2, the quotient will be less than the first digit by 2. What is the number?

Discuss the solutions of the following general problems:

5. What number, added to the denominators of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, will make the resulting fractions equal?

6. Having two kinds of wine worth a and b dollars a gallon, respectively, how many gallons of each kind must be taken to make a mixture of n gallons worth c dollars a gallon?

7. Two couriers, A and B, start at the same time from two stations, distant d miles from each other, and travel in the same direction. A travels n times as fast as B. Where will A overtake B?

CHAPTER XXV.

INFINITE SERIES.

1. The infinite series

$$1 + \tfrac{2}{3} + \tfrac{4}{9} + \tfrac{8}{27} + \cdots$$

is a decreasing geometrical progression, whose ratio is $\tfrac{2}{3}$.

Let $\qquad S_n = 1 + \tfrac{2}{3} + \tfrac{4}{9} + \tfrac{8}{27} + \cdots$ to n terms.

Then, by Ch. XXI., Art. 26,

$$S_n \doteq \frac{a}{1-r}, = \frac{1}{1-\frac{2}{3}}, = 3,$$

as n increases indefinitely. By actual computation, we obtain

$$S_1 = 1,\ S_2 = 1\tfrac{2}{3},\ S_3 = 2\tfrac{1}{9},\ S_4 = 2\tfrac{11}{27}, \text{ etc.}$$

These sums approach 3 more and more nearly, as more and more terms are included. This infinite series may therefore be regarded as having the finite sum 3.

But the sum of the series

$$1 + 2 + 4 + 8 + \cdots$$

increases beyond any finite number, however great, as the number of terms increases indefinitely.

2. The examples of the preceding article illustrate the following definitions:

An infinite series is said to be **Convergent**, when the sum of the first n terms approaches a definite finite limit, as n increases indefinitely.

An infinite series is said to be **Divergent**, when the sum of the first n terms increases numerically beyond any assigned number, however great, as n increases indefinitely.

3. It was shown in Ch. XXI., Art. 26, that the sum of n terms of the geometrical progression

$$a + ar + ar^2 + \cdots,$$

when $r < 1$, approaches the definite finite limit $\frac{a}{1-r}$, as n increases indefinitely.

Therefore, *any decreasing geometrical progression is a convergent series.*

4. Infinite series arise in connection with many mathematical operations. Thus, for example, if the division of 1 by $1-x$ be continued indefinitely, we obtain as a quotient the infinite series

$$1 + x + x^2 + x^3 + \cdots.$$

When x is numerically less than 1, that is, lies between -1 and 1, this series is a decreasing geometrical progression, as in Art. 1. Therefore, by the preceding article, it is convergent.

Thus, when $x = \frac{2}{3}$, as in Art. 1, the sum of n terms of the series approaches 3, as n increases indefinitely; and

$$\frac{1}{1-x} = \frac{1}{1-\frac{2}{3}} = 3.$$

Consequently, we may take the series as the expansion of the fraction, for all values of x between -1 and 1, and write

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots,$$

for such values of x.

When $x = 1$, the series becomes

$$1 + 1 + 1 + \cdots,$$

and is evidently divergent. If we assign as the value of the fraction, when $x = 1$, the limit of the fraction as x approaches 1, as in Ch. XXIV., Art. 19, we have

$$\frac{1}{1-x} = \infty,$$

when $x = 1$. Since both the fraction and the sum of the series are infinite when $x = 1$, in this sense we may assume that they are equivalent.

When $x = -1$, we have

$$1 - 1 + 1 - 1 + \cdots.$$

The sum of n terms of this series is 1 or 0, according as n is odd or even. The series is said to *oscillate*, and is neither convergent nor divergent. But, when $x = -1$,

$$\frac{1}{1-x} = \frac{1}{1+1} = \frac{1}{2}.$$

Consequently, we cannot assume that the series is the expansion of the fraction when $x = -1$.

When x is greater than 1, numerically, we have

$$S_n = \frac{1-x^n}{1-x}.$$

By taking n sufficiently great, x^n, and therefore $\frac{1-x^n}{1-x}$ can be made to exceed numerically any number, however great. Therefore, the series is divergent.

Thus, when $x = 2$, the series becomes

$$1 + 2 + 4 + 8 + \cdots.$$

But, when $x = 2$, $\frac{1}{1-x} = \frac{1}{1-2} = -1.$

Therefore, we cannot assume that the series is the expansion of the fraction when x is numerically greater than 1.

In general, an infinite series, no matter how obtained from a given expression, can be regarded as the expansion of the expression, for values of x which make the latter finite, *only when the series is convergent for such values of x.*

5. In the preceding article the convergency or divergency of the series was determined by an examination of the formula for the sum of n terms. There are, however, many infinite series for which such formulæ have not been obtained. In such cases it is necessary to determine the convergency or divergency of the series by other methods. Even when a formula for the sum of n terms is known, methods now to be given are often to be preferred.

6. In the theory which follows, we shall let S stand for the limit of the sum of n terms of the series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

as n increases indefinitely.

Also let $S_n = u_1 + u_2 + u_3 + \cdots + u_n$, the sum of n terms,

and $${}_mR_n = u_{n+1} + u_{n+2} + \cdots + u_{n+m},$$

the sum of m terms after the first n terms.

Then, evidently, $S_n + {}_mR_n = S_{n+m}$, and $\lim\limits_{n \doteq \infty} S_n = S$.

7. *The series*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is convergent if S_n remain finite, and ${}_mR_n$ approach 0 for all values of m, as n increases indefinitely; and, conversely, if the series be convergent, these two conditions are satisfied.

By the first condition S_n is finite. By the second condition,

$$\lim (S_{n+m} - S_n), = \lim {}_mR_n, = 0,$$

as n increases indefinitely.

Therefore, $$\lim S_{n+m} = \lim S_n.$$

That is, S_n cannot have one finite limit for one value of n, and a different finite limit for another value of n. Hence the limit of S_n is a *definite* finite number, and the series is convergent.

If, conversely, the series be convergent, the limit of S_n must be a *definite* finite number, and

$$\lim S_{n+m} = \lim S_n.$$

Hence $$\lim (S_{n+m} - S_n), = \lim {}_mR_n, = 0.$$

This principle is to be applied when it is possible to prove that the limit of the sum of n terms is finite, but not that it is a *definite* finite number. If, in addition, it can be proved that *lim* ${}_mR_n = 0$, this deficiency is supplied.

The oscillating series

$$1 - 1 + 1 - 1 + \cdots$$

is an instance of series which satisfy the first condition of the principle, but not the second. The limit of the sum of n terms is, as we have seen, 1 or 0, and is therefore finite.

Let $n = 2k$, an even number.

Then, $\lim (S_{2k+1} - S_{2k}), = {}_1R_{2k}, = 1$, not 0.

8. The following principle also does away with the necessity of proving that the limit of the sum of n terms is *definite* as well as finite, when the terms of the series are all positive.

If the sum of n terms of an infinite series of positive terms remain finite, as n increases indefinitely, the series is convergent.

For, since the sum continually increases, but remains finite, it must ultimately differ from some definite finite number by less than any assigned number, however small. This definite finite number is therefore the limit of the sum.

9. *If a finite number of terms be added to, or subtracted from, a given convergent series, the resulting series will be convergent; if added to, or subtracted from, a given divergent series, the resulting series will be divergent.*

For, the sum of a finite number of terms is a definite finite number. If this sum be added to the finite limit of the sum of n terms of a convergent series, the resulting sum will be a definite finite number, and the series therefore convergent.

In a similar manner the second part of the principle can be proved.

Methods of Comparison.

10. In the principles which now follow, it will be assumed that the terms of the series are all positive, unless the contrary is stated.

11. *If, after some particular term of an infinite series, each term be less than the corresponding term of a series known to be convergent, the given series is convergent.*

For, beginning with some particular term, which may or may not be the first term, the sum of n terms of the given series is less than the sum of the corresponding n terms of the known convergent series. This sum is therefore finite. Hence, by Art. 8, the series, beginning with some particular term, is convergent. It then follows from Art. 9, that the given series is convergent.

Ex. Compare the series

$$1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\cdots, =1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots,$$

with the known convergent series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots, =1+\frac{1}{2}+\frac{1}{2\cdot 2}+\frac{1}{2\cdot 2\cdot 2}+\cdots.$$

Evidently, *after the second term*, the denominator of each term of the given series is *greater* than the denominator of the corresponding term of the second series, and therefore each term of the given series, after the second, is *less* than the corresponding term of the second series.

Hence the given series is convergent.

12. *If, after some particular term of a given infinite series, each term be greater than the corresponding term of a known divergent series, the given series is divergent.*

The proof of this principle is similar to that of the preceding article.

Ex. Compare the series $\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\cdots$ with the known divergent series $1+1+1+\cdots$.

Each term of the given series is greater than the corresponding term of the second series. Hence the given series is *divergent*.

13. In applying the principles of Arts. 11-12, certain series are important. These we now discuss.

(i.) Examine the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots.$$

We have
$$1+\frac{1}{2}=1+\frac{1}{2};$$
$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4},=\frac{1}{2};$$
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8},=\frac{1}{2};$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Whence, by addition,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

The series in the second member is evidently divergent, and hence, with greater reason, the given series is divergent.

(ii.) The preceding series is a particular instance of the series

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots$$

wherein k is assumed to be positive.

We will first show for what values of k this series is convergent, by comparing it with a series of greater terms.

$$1=1;$$
$$\frac{1}{2^k}+\frac{1}{3^k}<\frac{1}{2^k}+\frac{1}{2^k},=\frac{2}{2^k},=\frac{1}{2^{k-1}};$$
$$\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}<\frac{1}{4^k}+\frac{1}{4^k}+\frac{1}{4^k}+\frac{1}{4^k},=\frac{1}{4^{k-1}};$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Whence, by addition,

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots<1+\frac{1}{2^{k-1}}+\frac{1}{4^{k-1}}+\frac{1}{8^{k-1}}\cdots.$$

The series in the second member is a geometrical progression whose ratio is $(\frac{1}{2})^{k-1}$. When $k > 1$, $k-1$ is positive, integral or fractional. Therefore $(\frac{1}{2})^{k-1}$ is less than 1, and the series in the second member is convergent, by Art. 3. Consequently, by Art. 8, the given series is convergent, when $k > 1$.

E.g., the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ is convergent.

When $k = 1$, the series is that which was proved to be divergent in (i.).

When $k < 1$, $2^k < 2$, and therefore $\frac{1}{2^k} > \frac{1}{2}$.

In like manner, $\frac{1}{3^k} > \frac{1}{3}, \; \frac{1}{4^k} > \frac{1}{4}, \cdots.$

Therefore,

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

Since the series in the second member is divergent, the given series is divergent, when $k < 1$.

E.g., $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots, \; = 1 + \frac{1}{2^{\frac{1}{2}}} + \frac{1}{3^{\frac{1}{2}}} + \frac{1}{4^{\frac{1}{2}}} + \cdots,$
is a divergent series.

We therefore conclude:

The series $$1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$$

is convergent when $k > 1$, and divergent when $k = 1$, or $k < 1$.

14. In thus comparing one series with another it is important not to be misled by the relative values of the first few terms of the two series.

Thus, compare the terms of the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots$$

with the corresponding terms of the series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots.$$

We have $\frac{1}{2\cdot 3}<\frac{1}{2^2}$, $\frac{1}{3\cdot 4}<\frac{1}{2^3}$, $\frac{1}{4\cdot 5}<\frac{1}{2^4}$,

but $\frac{1}{5\cdot 6}>\frac{1}{2^5}$; and so for all subsequent terms. Had we given attention only to the first four terms, we should have inferred that the first series is convergent from a comparison with the second series, whereas the question of its convergency or divergency is not settled. Now compare the given series with the known convergent series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots, \quad =1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}+\frac{1}{4\cdot 4}+\cdots.$$

It is evident, from the forms of the denominators in the two series, that each term of the given series, after the first, is less than the corresponding term of the last series. Hence the given series is convergent.

EXERCISES I.

Determine the convergency or divergency of the series:

1. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\cdots.$

2. $\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{2}{3}}+\sqrt{\tfrac{3}{4}}+\cdots.$

3. $\frac{3}{1\cdot 2}+\frac{4}{2\cdot 3}+\frac{5}{3\cdot 4}+\cdots.$

4. $\frac{2}{1\cdot 3}+\frac{3}{2^2\cdot 5}+\frac{4}{3^2\cdot 7}+\cdots.$

5. $\frac{2}{3}+\frac{2\cdot 3}{3\cdot 5}+\frac{2\cdot 3\cdot 4}{3\cdot 5\cdot 7}+\cdots.$

6. $\frac{2}{1\cdot 3}+\frac{3}{2\cdot 4}+\frac{4}{3\cdot 5}+\cdots.$

7. $\frac{2}{1}+\frac{3}{2^2}+\frac{4}{3^2}+\cdots.$

8. $1+\frac{3}{\lfloor 2}+\frac{3^2}{\lfloor 3}+\cdots.$

9. $\frac{1}{1+2}+\frac{1}{1+2^2}+\frac{1}{1+2^3}+\cdots.$

10. $\frac{a+x}{b+x}+\frac{(a+x)(2a+x)}{(b+x)(2b+x)}+\frac{(a+x)(2a+x)(3a+x)}{(b+x)(2b+x)(3b+x)}+\cdots,$

wherein a, b, and x are positive, and $b>a$.

The General Term of a Series.

15. If the general term, the nth say, be given, the entire series is known.

Ex. Write the series whose nth term is

$$\frac{2^{n-1}}{(2n-1)(2n+1)}\cdot\frac{x^{4n+1}}{4n+1}.$$

Giving to n the series of values 1, 2, 3, ⋯, we obtain

$$\frac{1}{1\cdot 3}\cdot\frac{x^5}{5}+\frac{2}{3\cdot 5}\cdot\frac{x^9}{9}+\frac{2^2}{5\cdot 7}\cdot\frac{x^{13}}{13}+\cdots.$$

16. It is frequently necessary to write the nth term of a series, when only the first few terms are given.

Ex. Write the nth term of the series

$$1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\cdots.$$

The exponent of x in the second term is $2\times 2-2, =2$; in the third term, $2\times 3-2, =4$; and in the nth, $2n-2$.

The first factor in the numerator of each term after the first is 1, and the last is one less than the exponent of x. Hence the numerator of the nth term is $1\cdot 3\cdot 5\cdots(2n-3)$.

In the denominators, after the first term, the first factor is 2, and the last is the same as the exponent of x. Hence the denominator of the nth term is $2\cdot 4\cdot 6\cdots(2n-2)$.

Therefore, the nth term is $\dfrac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}x^{2n-2}$.

Observe that only the terms after the first are obtained from the nth term as thus written.

17. *If the ratio of each term of a given infinite series to the corresponding term of another infinite series be finite, the given series is convergent when the second series is convergent, and is divergent when the second series is divergent.*

Let $$u_1+u_2+u_3+\cdots+u_n+\cdots$$
be the given series,

and $$v_1+v_2+v_3+\cdots+v_n+\cdots$$
be a series known to be convergent, or divergent.

First, let the second series be convergent.

Let k be a finite number greater than the greatest of the finite ratios

$$\frac{u_1}{v_1}, \frac{u_2}{v_2}, \cdots, \frac{u_n}{v_n}, \cdots.$$

Then
$$\frac{u_1}{v_1} < k, \frac{u_2}{v_2} < k, \frac{u_3}{v_3} < k, \cdots, \frac{u_n}{v_n} < k, \cdots;$$

whence, $u_1 < kv_1$, $u_2 < kv_2$, $u_3 < kv_3$, $\cdots$, $u_n < kv_n$, $\cdots$.

By addition, we have

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots < k\,(v_1 + v_2 + v_3 + \cdots + v_n + \cdots).$$

Since $v_1 + v_2 + \cdots + v_n$ remains finite as n increases indefinitely, and k is finite, $u_1 + u_2 + u_3 + \cdots + u_n$ remains finite, as n increases indefinitely. Hence, by Art. 8, the given series is convergent.

Next, let the second series be divergent.

Now let k be a finite number less than the least of the finite ratios

$$\frac{u_1}{v_1}, \frac{u_2}{v_2}, \cdots, \frac{u_n}{v_n}, \cdots.$$

Then, we readily obtain

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots > k(v_1 + v_2 + v_3 + \cdots + v_n + \cdots).$$

Since the second series is divergent, $v_1 + v_2 + v_3 + \cdots + v_n$, and with greater reason, $u_1 + u_2 + u_3 + \cdots + u_n$, increases beyond any assigned number, however great, as n increases indefinitely.

Therefore the given series is divergent.

Ex. 1. Examine the series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4\cdot} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} + \cdots.$$

Compare this series with the known convergent series [Art. 13 (ii.)]

$$\frac{1}{1} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \cdots.$$

It is sufficient to examine the ratio of the nth term of the given series to the nth term of the second series.

This ratio is

$$\frac{1}{n\,(n+1)(n+2)} \div \frac{1}{n^3}, = \frac{n^2}{(n+1)(n+2)}, \doteq 1,$$

as n increases indefinitely. Since, therefore, the ratio is always finite, the given series is convergent.

Ex. 2. Examine the series

$$\frac{2}{1\cdot 3} + \frac{4}{3\cdot 5} + \frac{6}{5\cdot 7} + \cdots + \frac{2\,n}{(2\,n-1)(2\,n+1)} + \cdots.$$

Compare with the known divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

The ratio of the nth terms is

$$\frac{2\,n}{(2\,n-1)(2\,n+1)} \div \frac{1}{n}, = \frac{2\,n^2}{(2\,n-1)(2\,n+1)}, \doteq \frac{1}{2},$$

as n increases indefinitely.

Since this ratio is finite for all values of n, the given series is divergent.

EXERCISES II.

Determine the convergency or divergency of the series whose nth terms are:

1. $\dfrac{2\,n-5}{n^3-5\,n}$.

2. $\dfrac{1+n}{1+n^2}$.

3. $\dfrac{n+2}{n^3+1}$.

4. $\dfrac{n^2-(n-1)^2}{n^2+(n+1)^2}$.

5. $\dfrac{(n+a)(n+b)}{n(n+1)(n+2)}$.

Determine the convergency or divergency of the series:

6. $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \cdots$.

7. $\dfrac{1}{2} + \dfrac{2}{3\sqrt{2}} + \dfrac{3}{4\sqrt{3}} + \cdots$.

8. $\dfrac{3}{1\cdot 2} + \dfrac{5}{2^2\cdot 3} + \dfrac{7}{3^2\cdot 4} + \cdots$.

9. $\dfrac{8}{2\cdot 3} + \dfrac{16}{3\cdot 4} + \dfrac{25}{4\cdot 5} + \cdots$.

10. $\dfrac{1}{a(a+b)} + \dfrac{1}{(a+2\,b)(a+3\,b)} + \dfrac{1}{(a+4\,b)(a+5\,b)} + \cdots$.

Series having Negative Terms.

18. *If a series be convergent when its terms are all positive, it will remain convergent when some or all of its terms are made negative.*

Since S_n remains finite and ${}_mR_n \doteq 0$, when all the terms are positive, with greater reason S_n will remain finite and ${}_mR_n \doteq 0$, when some or all of the terms are made negative.

Ex. The series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$ is convergent,
since the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is convergent.

19. A series which is convergent when all its negative terms are made positive is said to be **Absolutely Convergent.**

Evidently every convergent series whose terms are all positive is absolutely convergent.

20. *If the terms of an infinite series be alternately positive and negative, and the nth term approach 0, as n increases indefinitely, the series is convergent.*

Let the given series be

$$u_1 - u_2 + u_3 - \cdots + (-1)^{n-1}u_n + \cdots.$$

Then
$$
\begin{aligned}
S_n &= u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n \\
&= (u_1 - u_2) + (u_3 - u_4) + \cdots \qquad (1) \\
&= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots. \qquad (2)
\end{aligned}
$$

Since the terms decrease numerically, it is evident that in (1) and (2) the sums inclosed in the parentheses are positive. Therefore from (1) we infer that S_n is *positive*, and from (2) that it is *less than the first term* u_1. Therefore S_n is finite.

Also,

$$
\begin{aligned}
{}_mR_n &= (-1)^n[u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} + \cdots + (-1)^{m-1}u_{n+m}] \\
&= (-1)^n[(u_{n+1} - u_{n+2}) + (u_{n+3} - u_{n+4}) + \cdots] \qquad (3) \\
&= (-1)^n[u_{n+1} - (u_{n+2} - u_{n+3}) - \cdots]. \qquad (4)
\end{aligned}
$$

From (3) we infer that the part of ${}_mR_n$ in the brackets is positive, and from (4) that it is less than u_{n+1}. Since $u_{n+1} \doteq 0$, it follows that ${}_nR_n \doteq 0$. Hence the given series is convergent.

Ex. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, but not absolutely convergent, since $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.

The Ratio of Convergency.

21. In the following principle the terms of the series are not necessarily all positive:

An infinite series is convergent, if, after some particular term, the ratio of each term to the preceding be numerically less than some fixed positive number, which is itself less than unity.

Let the given series be

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

and let the ratio of each term after the kth to the preceding be less than r, which is itself less than 1.

First, assume that the terms are all positive.

Then, from $\dfrac{u_{k+1}}{u_k} < r,\ \dfrac{u_{k+2}}{u_{k+1}} < r,\ \dfrac{u_{k+3}}{u_{k+2}} < r,\ \cdots;$

we obtain

$$u_{k+1} < ru_k,\ u_{k+2} < ru_{k+1} < r^2u_k,\ u_{k+3} < ru_{k+2} < r^3u_k,\ \cdots.$$

Whence, by addition,

$$u_{k+1} + u_{k+2} + u_{k+3} + \cdots < u_k(r + r^2 + r^3 + \cdots).$$

Since $r < 1$, $r + r^2 + r^3 + \cdots \doteq \dfrac{r}{1-r}$, a finite number.

Therefore, since the sum of the finite number of terms $u_1 + u_2 + \cdots + u_{k-1}$ is finite, the given series is, by Art. 8, convergent.

When some or all of the terms are negative, the series is, by *Art.* 18, convergent.

22. *An infinite series of positive terms is divergent, if, after some particular term, the ratio of each term to the preceding be equal to unity, or greater than unity.*

In the series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

let the ratio of each term after the kth be equal to 1. Then the sum of n terms, after the kth, is equal to nu_k, and hence increases beyond any assigned number however great, as n increases indefinitely.

Next, let this ratio be greater than unity; then the sum of n terms after the kth is greater than nu_k, and hence increases beyond any assigned number, however great, as n increases indefinitely.

Therefore, in each case, the series is divergent.

23. The ratio of the nth term to the preceding is called the **Ratio of Convergency** of the series.

24. The following examples will illustrate the principles of Arts. 21–22.

Ex. 1. Examine the series

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} + \cdots.$$

The ratio of convergency is

$$\frac{n}{2^n} \div \frac{n-1}{2^{n-1}}, \quad = \frac{n}{n-1} \cdot \frac{1}{2}, \quad = \frac{1}{1 - \frac{1}{n}} \cdot \frac{1}{2}, \quad \doteq \frac{1}{2}.$$

By taking n large enough we can make this ratio differ from $\frac{1}{2}$ by as little as we please, and consequently less than some number between $\frac{1}{2}$ and 1; that is, less than some number which is itself less than 1.

Thus, if $n = 4$, the ratio is equal to $\frac{2}{3}$, which is less than, say $\frac{3}{4}$. That the ratio will remain less than $\frac{3}{4}$ for values of n greater than 4, can be shown as follows.

Assume $\frac{n}{n-1}\cdot\frac{1}{2}<\frac{3}{4}$; then $2n<3n-3$, and $n>3$.

Since, therefore, after the third term, the ratio of each term to the preceding is less than $\frac{3}{4}$, which is less than 1, the given series is convergent.

Ex. 2. Examine the series

$$\frac{1\cdot 3}{2}+\frac{3\cdot 5}{2^2}x+\frac{5\cdot 7}{2^3}x^2+\cdots\frac{(2n-1)(2n+1)}{2^n}x^{n-1}+\cdots.$$

The ratio of convergency is

$$\frac{(2n-1)(2n+1)}{2^n}x^{n-1}\div\frac{(2n-3)(2n-1)}{2^{n-1}}x^{n-2}$$
$$=\frac{(2n+1)x}{(2n-3)2},\doteq\frac{x}{2}.$$

By taking n sufficiently great, we can make this ratio differ from $\frac{1}{2}x$ by as little as we please. If, therefore, x have a definite value less than 2, the ratio can be made less than some number, say k, which is itself less than 1. Hence the series is convergent when $x<2$.

The term after which this ratio becomes and remains less than k is determined from

$$\frac{2n+1}{2n-3}\cdot\frac{x}{2}<k,\ \text{whence}\ n>\frac{6k+x}{2(2k-x)}.$$

Thus, let $x=\frac{3}{2}$, or $\frac{1}{2}x=\frac{3}{4}$, and $k=\frac{5}{6}$. We find $n>19\frac{1}{2}$. That is, when $x=\frac{3}{2}$, the ratio of each term, after the 19th, to the preceding is less than $\frac{5}{6}$, which is less than 1.

Evidently, when $x=2$, or $x>2$, the ratio is greater than 1 for all values of n. Therefore the series is then divergent.

25. The significance of the words, *less than some number which is itself less than unity*, is shown by an examination of the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots,$$

which is known to be divergent.

The ratio of convergency is $\frac{n-1}{n}, = 1 - \frac{1}{n}.$

This ratio is less than 1, but by taking n large enough, it can be made to differ from 1 by as little as we please. The value of this ratio will therefore not remain less than some definite number, which is itself less than 1. This condition of the principle of Art. 21 is not satisfied, and the test fails. Also, since neither condition of Art. 22 is satisfied, the test fails to prove the series divergent. In such cases, it is necessary to try other tests, just as the above series was by other means proved to be divergent.

26. It is not, in general, necessary to determine the number of the term after which the ratio of any term to the preceding is less than some definite number which is itself less than 1, in the case of a convergent series. The following method, illustrated by the examples of the preceding articles, is sufficient:

Determine the limit of the ratio of convergency as n increases indefinitely.

(i.) *If this limit* < 1, *the series is convergent.*

(ii.) *If this limit* > 1, *the series is divergent.*

(iii.) *If this limit* $= 1$, *the convergency or divergency of the series is, as a rule, not settled, and some other test must be applied.*

But, if the ratio be always greater than 1, *as it approaches the limit* 1, *the series is, by Art.* 22, *divergent.*

Ex. Examine the series

$$\frac{4 \cdot 5\,x}{1 \cdot 2 \cdot 3} + \frac{5 \cdot 6\,x^2}{2 \cdot 3 \cdot 4} + \cdots + \frac{(n+3)(n+4)x^n}{n(n+1)(n+2)} + \cdots.$$

The ratio of convergency is

$$\frac{(n+3)(n+4)x^n}{n(n+1)(n+2)} \div \frac{(n+2)(n+3)x^{n-1}}{(n-1)n(n+1)},$$

$$= \frac{(n+4)(n-1)}{(n+2)(n+2)} \cdot x, \ = x,$$

as n increases indefinitely.

Hence, for values of $x < 1$, the series is convergent; for values of $x > 1$, the series is divergent; while, for $x = 1$, the series is in doubt. When $x = 1$, we have

$$\frac{4 \cdot 5}{1 \cdot 2 \cdot 3} + \frac{5 \cdot 6}{2 \cdot 3 \cdot 4} + \cdots + \frac{(n+3)(n+4)}{n(n+1)(n+2)} + \cdots.$$

We will try the method of Art. 17, comparing with the known divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$$

The ratio of the nth term of the given series to the nth term of the auxiliary series is

$$\frac{(n+3)(n+4)}{n(n+1)(n+2)} \div \frac{1}{n}, = \frac{(n+3)(n+4)}{(n+1)(n+2)}, \doteq 1.$$

This ratio is evidently finite for all values of n. Therefore, when $x = 1$, the given series is divergent.

27. The following application of the principle of Art. 21 will be required in Ch. XXVII.

The series

$$1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots$$

is absolutely convergent, when <1 numerically.

In the above series n is finite. We will therefore take the ratio of the $(k+1)$th term to the preceding.

The ratio of convergence is

$$\frac{n(n-1)\cdots(n-k+1)}{\underline{|k}}x^k \div \frac{n(n-1)\cdots(n-k+2)}{\underline{|k-1}}x^{k-1}$$

$$= \frac{n-k+1}{k}x, \doteq -x,$$

as k increases indefinitely.

Therefore, the series is absolutely convergent, when <1 numerically.

EXERCISES III.

Determine the convergency or divergency of the series:

1. $1+\frac{2^k}{\lfloor\underline{2}}+\frac{3^k}{\lfloor\underline{3}}+\cdots.$

2. $\frac{2}{1}+\frac{2\cdot 3}{1\cdot 3}+\frac{2\cdot 3\cdot 4}{1\cdot 3\cdot 5}+\cdots.$

3. $\frac{1}{4}+\frac{1\cdot 3}{4\cdot 7}+\frac{1\cdot 3\cdot 5}{4\cdot 7\cdot 10}+\cdots.$

4. $\frac{1}{3}+\frac{1\cdot 2}{3\cdot 6}+\frac{1\cdot 2\cdot 3}{3\cdot 6\cdot 9}+\cdots.$

5. $\frac{1}{a+1}+\frac{k}{a+k}+\frac{k^2}{a+2k}+\cdots.$

Determine for what values of x the following series are convergent or divergent:

6. $1^2+2^2x+3^2x^2+\cdots.$

7. $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots.$

8. $1+\frac{x}{\lfloor\underline{1}}+\frac{x^2}{\lfloor\underline{2}}+\cdots.$

9. $\frac{1}{1\cdot 2}+\frac{x}{2\cdot 3}+\frac{x^2}{3\cdot 4}+\cdots.$

10. $\frac{\pi}{1}-\frac{1}{x}+\frac{1}{3x^3}-\cdots.$

11. $\frac{1}{1\cdot 3}+\frac{2x}{3\cdot 5}+\frac{(2x)^2}{5\cdot 7}+\cdots.$

12. $1-\frac{3x}{2^2}+\frac{5x^2}{3^2}-\cdots.$

13. $1+\frac{4x}{5}+\frac{9x^2}{5^2}+\cdots.$

14. $1+\frac{3^3x}{\lfloor\underline{2}}+\frac{5^3x^2}{\lfloor\underline{3}}+\cdots.$

15. $1+2^2x+\frac{3^2x^2}{\lfloor\underline{2}}+\cdots.$

16. $a+(a+d)x+(a+2d)x^2+\cdots.$

17. $\frac{1}{1+x}+\frac{1}{1+x^2}+\frac{1}{1+x^3}+\cdots.$

18. $\frac{1}{1+x}+\frac{x}{1+x^2}+\frac{x^2}{1+x^4}+\cdots.$

19. $\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\cdots.$

20. $1-\frac{x}{1+k}+\frac{x^2}{1+2k}-\cdots.$

CHAPTER XXVI.

UNDETERMINED COEFFICIENTS.

1. Upon the following principles is based an important method of changing an algebraical expression from one form to another.

2. *If an infinite series* $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ *be convergent for values of* x *greater than* 0, *the sum of the series approaches* a_0, *as* x *approaches* 0.

Let $$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = a_0 + xS_1,$$
wherein $$S_1 = a_1 + a_2x + a_3x^2 + \cdots.$$

Evidently, if the given series be convergent, that is, if $a_0 + xS_1$ be finite, then S_1 is finite. Therefore, by Ch. XXIV., Art. 16, $xS_1 \doteq 0$, when $x \doteq 0$.

Consequently
$$a_0 + a_1x + a_2x^2 + \cdots, = a_0 + xS, \doteq a_0, \text{ when } x \doteq 0.$$

3. *If two integral series, arranged to ascending powers of* x, *be equal for all values of* x *which make them both convergent, the coefficients of like powers of* x *are equal.*

Let $$a_0 + a_1x + a_2x^2 + \cdots = b_0 + b_1x + b_2x^2 + \cdots$$
for all values of x which make the two series convergent.

Then the sums of the two series approach equal limits when $x \doteq 0$. But, by the preceding article, the sum of the one series approaches a_0, that of the other b_0; consequently $a_0 = b_0$,

and $$a_1x + a_2x^2 + \cdots = b_1x + b_2x^2 + \cdots.$$

Since by Ch. XXV, Art. 21, these two series are convergent for all values of x for which the original series are convergent, they are equal for values of x other than zero, and the last equation may be divided by x.

Hence $a_1 + a_2x + a_3x^2 + \cdots = b_1 + b_2x + b_3x^2 + \cdots$;

and as before, $a_1 = b_1,$

and $a_2x + a_3x^2 + \cdots = b_2x + b_3x^2 + \cdots.$

In like manner, we can prove $a_2 = b_2$, $a_3 = b_3$, etc.

4. The principle of Art. 3 holds with greater reason if either or both of the series be finite. The series must be equal for all values of x, if they be both finite; or, if one be infinite, for all values of x which make that series convergent.

5. The condition that the roots of the equation

$$ax^2 + bx + c = 0$$

are equal, given in Ch. XVIII., Art. 12 (ii.), can be obtained also by applying the principle of Art. 3.

If the two roots be equal, $ax^2 + bx + c$ is the square of a binomial. We therefore assume

$$ax^2 + bx + c = (Ax + B)^2 = A^2x^2 + 2\,ABx + B^2.$$

By Art. 3, $A^2 = a$ (1), $2\,AB = b$ (2), $B^2 = c$ (3).

From (1) and (3), $A = \sqrt{a},\ B = \sqrt{c}.$

Whence, by (2), $2\sqrt{(ac)} = b$, or $b^2 = 4\,ac$.

Expansion of Rational Fractions.

6. We shall now give a method of expanding a fraction in an infinite series, without performing the actual division.

Ex. 1. Expand $\dfrac{2-x}{1+x-x^2},$

in a series, to ascending powers of x.

We equate the fraction to a series of the required form, in which the coefficients of the different powers of x are unknown, or *undetermined*.

Assume $\dfrac{2-x}{1+x-x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots,$

wherein $A, B, C, D, E, \cdots$ are constants to be determined.

Clearing the equation of fractions, we obtain

$$\begin{array}{r|r|r|r|l}
2-x=A+B & x+C & x^2+D & x^3+E & x^4+\cdots. \\
A & +B & +C & +D & \\
 & -A & -B & -C &
\end{array}$$

In this work the powers of x in the terms of the second and third partial products are omitted, it being understood that the letters remaining are the coefficients of the powers of x just above in the first partial product.

Thus the coefficient of x is $A+B$, etc.

The series on the right is infinite; that on the left may be regarded as an infinite series with zero coefficients of all powers of x higher than the first. By Art. 3, we have

$$\begin{array}{lll}
A=2; & B+A=-1, & \text{whence } B=-3; \\
 & C+B-A=0, & \text{whence } C=5; \\
 & D+C-B=0, & \text{whence } D=-8; \\
 & E+D-C=0, & \text{whence } E=13; \\
 & \text{etc.,} & \text{etc.}
\end{array}$$

Hence, substituting these values of $A, B, C, D, \cdots$ in the assumed series, we have

$$\frac{2-x}{1+x-x^2}=2-3x+5x^2-8x^3+13x^4+\cdots.$$

We can assume that this series is equivalent to the fraction only when x has such values as make it convergent.

Let the student compare this result with that obtained by division. In fact, the latter method of expanding a fraction is to be preferred when only a few terms are wanted. But the successive coefficients, after a certain stage, may be computed with great facility by the *method of undetermined coefficients*. A moment's inspection of the preceding work will convince the student that the coefficient D, and all which follow it, are each connected with the two immediately preceding coefficients by a definite relation. Thus,

$$D+C-B=0,\ E+D-C=0,\ F+E-D=0, \text{ etc.}$$

In assuming as the expansion of a rational fraction an infinite series of ascending powers of x, it is usually necessary first to determine with what power the series should commence. This is done by division, when both numerator and denominator are arranged to ascending powers of x. In fact, this step also determines completely the first term of the series.

Ex. 2. Expand $$\frac{1-x}{3x^2-x^3},$$

in a series to ascending powers of x.

The first term in the expansion, obtained by division, is evidently $\frac{1}{3}x^{-2}$.

We therefore assume

$$\frac{1-x}{3x^2-x^3} = \frac{1}{3}x^{-2} + Bx^{-1} + C + Dx + Ex^2 + Fx^3 + \cdots.$$

Clearing of fractions, we obtain

$$1 - x = 1 + \left.\begin{matrix} 3B \\ -\frac{1}{3} \end{matrix}\right| x + \left.\begin{matrix} 3C \\ -B \end{matrix}\right| x^2 + \left.\begin{matrix} 3D \\ -C \end{matrix}\right| x^3 + \begin{matrix} \cdots. \\ -\cdots. \end{matrix}$$

By Art. 3, we have

$$\begin{aligned} 1 = 1, \quad 3B - \tfrac{1}{3} &= -1, \text{ whence } B = -\tfrac{2}{9}; \\ 3C - B &= 0, \text{ whence } C = -\tfrac{2}{27}; \\ 3D - C &= 0, \text{ whence } D = -\tfrac{2}{81}; \\ \text{etc.,} & \qquad\qquad \text{etc.} \end{aligned}$$

Hence, $$\frac{1-x}{3x^2-x^3} = \frac{1}{3}x^{-2} - \frac{2}{9}x^{-1} - \frac{2}{27} - \frac{2}{81}x - \cdots.$$

EXERCISES I.

Expand the following fractions in series, to ascending powers of x, to four terms:

1. $\dfrac{1}{1-2x}$.
2. $\dfrac{3}{1+3x}$.
3. $\dfrac{6}{3-x}$.
4. $\dfrac{1+x}{1-x}$.
5. $\dfrac{2-5x}{1+2x}$.
6. $\dfrac{3x+x^2}{1-2x}$.
7. $\dfrac{x^3-3x^2}{x^2-2}$.
8. $\dfrac{1-x}{5x^2+2x^3}$
9. $\dfrac{1}{1+x+x^2}$.

10. $\dfrac{1+2x}{1+x-x^2}$. 11. $\dfrac{2-x}{1+2x-3x^2}$. 12. $\dfrac{3-2x^2}{2-3x+x^2}$.

13. $\dfrac{2+x-3x^2}{3-x+3x^2}$. 14. $\dfrac{x^4-3x^2+1}{1+x-2x^2}$. 15. $\dfrac{1}{2x^2-6x^3+x^4}$.

Expansion of Surds.

7. Ex. Expand $\sqrt{(1-x^2+2x^3)}$,

in a series, to ascending powers of x. Assume

$$\sqrt{(1-x^2+2x^3)}=1+Bx+Cx^2+Dx^3+Ex^4+\cdots.$$

Squaring both sides of the equation, we have

$$\begin{array}{l|l|l|l|l}
1-x^2+2x^3=1+2B & x+2C & x^2+2D & x^3+2E & x^4+\cdots, \\
 & +B^2 & +2BC & +2BD & \cdots. \\
 & & & +C^2 & \cdots.
\end{array}$$

Equating coefficients, $1=1$.

$$\begin{aligned}
2B&=0, &&\text{whence } B=0;\\
2C+B^2&=-1, &&\text{whence } C=-\tfrac{1}{2};\\
2D+2BC&=2, &&\text{whence } D=+1;\\
2E+2BD+C^2&=0, &&\text{whence } E=-\tfrac{1}{8};\ \text{etc.}
\end{aligned}$$

Hence $\sqrt{(1-x^2+2x^3)}=1-\frac{1}{2}x^2+x^3-\frac{1}{8}x^4+\cdots$

EXERCISES II.

Expand the following expressions in series, to ascending powers of x, to four terms:

1. $\sqrt{(1+x)}$. 2. $\sqrt{(a^2-2x^2)}$. 3. $\sqrt[3]{(1-x^2)}$.

4. $\sqrt{(4-2x+x^2)}$. 5. $\sqrt{(5+3x+9x^2)}$. 6. $\sqrt[3]{(1-x+x^2)}$.

Partial Fractions.

8. It is frequently desirable to separate a rational algebraical fraction into the simpler (*partial*) fractions of which it is the algebraical sum.

E.g., $$\frac{2x}{1-x^2}=\frac{1}{1-x}-\frac{1}{1+x}.$$

The process of separating a given fraction into its partial fractions is, therefore, the converse of addition (including subtraction) of fractions; and this fact must guide us in assuming the forms of the partial fractions.

We shall also assume that the degree of the numerator is at least one less than that of the denominator. A fraction whose numerator is of a degree equal to or greater than that of its denominator can be first reduced by division to the sum of an integral expression and a fraction satisfying the above condition. The latter fraction will then be decomposed.

The *denominators* of the partial fractions can be definitely assumed. For they are evidently those factors whose lowest common multiple is the denominator of the given fraction. But there is one case of doubt; namely, when a prime factor is repeated in the denominator of the given fraction.

E.g.,

$$\frac{6-2x^2}{(1-x)^2(1+x)}=\frac{3}{1-x}+\frac{2}{(1-x)^2}+\frac{1}{1+x};$$

$$\frac{3+x^2}{(1-x)^2(1+x)}=\frac{2}{(1-x)^2}+\frac{1}{1+x}.$$

We could not have decided, in advance, whether either of the two given fractions is the sum of two or of three partial fractions. There must necessarily be a partial fraction having $(1-x)^2$ as a denominator, since, otherwise, the L. C. M. of the denominators would not contain the prime factor $1-x$ to the second power. But it cannot be determined, in advance, whether there is a partial fraction having $1-x$ as a denominator.

In such cases, therefore, it is advisable to make provision for all possible partial fractions by assuming as denominators all repeated factors to the first power, second power, etc.

The numerators of partial fractions thereby assumed, which should not have been included, will acquire the value zero from the subsequent work, so that those fractions drop out of the result.

The *numerators* of the partial fractions must be assumed with undetermined coefficients. Since the numerator of the given fraction is, by the hypothesis, of degree at least one less than the denominator, the same must be true of each partial fraction. We therefore assume, for each numerator, a *complete* rational integral expression with undetermined coefficients of degree one lower than the corresponding denominator.

If any term in the assumed form of the numerator should not have been included, its coefficient will prove to be zero.

An exception to this principle occurs when the denominator of the partial fraction is the second or higher power of a prime factor, as, $(1-x)^2$. In that case the numerator is assumed as it would be according to the above principle if the prime factor occurred to the first power only.

We may briefly restate the above principles:

Separate the denominator of the given fraction into its prime factors. Assume as the denominator of a partial fraction each prime factor; in particular, when a prime factor enters to the nth power, assume that factor to the first power, second power, and so on, to the nth power, as a denominator.

Assume for each numerator a rational integral expression, with undetermined coefficients, of degree one lower than the prime factor in the corresponding denominator.

Let us first decompose the two fractions which we have used to illustrate the theory.

Ex. 1. $$\frac{6-2x^2}{(1-x)^2(1+x)}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{1+x}.$$

Since the prime factor in the denominator of each partial fraction is of the first degree, each numerator is assumed to be of the zeroth degree.

Clearing the equation of fractions, we have

$$\begin{aligned}6-2x^2&=A(1-x)(1+x)+B(1+x)+C(1-x)^2\\&=(-A+C)x^2+(B-2C)x+A+B+C.\end{aligned}$$

Since this equation must be true for all values of x, we have

$$\left.\begin{aligned} -A+C&=-2,\\ B-2C&=0,\\ A+B+C&=6.\end{aligned}\right\}\text{Whence } A=3,\ B=2,\ C=1.$$

Ex. 2. $\dfrac{3+x^2}{(1-x)^2(1+x)}=\dfrac{A}{1-x}+\dfrac{B}{(1-x)^2}+\dfrac{C}{1+x}.$

The forms of the partial fractions are assumed the same as in Ex. 1. We have

$$3+x^2=(-A+C)x^2+(B-2C)x+A+B+C,$$

and then
$$\left.\begin{aligned} -A+C&=1,\\ B-2C&=0,\\ A+B+C&=3.\end{aligned}\right\}\text{Whence } A=0,\ B=2,\ C=1.$$

Therefore $\dfrac{3+x^2}{(1-x)^2(1+x)}=\dfrac{2}{(1-x)^2}+\dfrac{1}{1+x}.$

When the factors of the denominator of the given fraction are of the first degree, as in Exs. 1 and 2, the work may be shortened.

Begin with the equation

$$6-2x^2=A(1-x)(1+x)+B(1+x)+C(1-x)^2,$$

of Ex. 1. Since this equation is true for all values of x, we may substitute in it for x any value we please. Let us take such a value as will make one of the prime factors zero.

Substituting 1 for x, we obtain

$$4=2B,\ \text{whence } B=2.$$

Next, letting $x=-1$, we have

$$4=4C,\ \text{whence } C=1.$$

There is no other value of x which will make a prime factor zero, but any other value, the smaller the better, will give an equation in which we may substitute the values of B and C already obtained.

Letting $x = 0$, we obtain

$$6 = A + B + C, \text{ whence } A = 3.$$

The same method can be applied to Ex. 2.

Ex. 3. $$\frac{x^2 - x + 3}{x^3 - 1} = \frac{x^2 - x + 3}{(x-1)(x^2 + x + 1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}.$$

In this example, the one prime factor being of the second degree, we assume the corresponding numerator to be a complete linear expression.

Clearing of fractions, we have

$$x^2 - x + 3 = A(x^2 + x + 1) + (Bx + C)(x - 1) =$$
$$(A + B)x^2 + (A - B + C)x + A - C.$$

Equating coefficients of like powers of x, we obtain

$$A + B = 1,\ A - B + C = -1,\ A - C = 3;$$

whence, $$A = 1,\ B = 0,\ C = -2.$$

Or, we might have used the second method, beginning with

$$x^2 - x + 3 = A(x^2 + x + 1) + (Bx + C)(x - 1).$$

Letting $x = 1$, we obtain

$$3 = 3A, \text{ whence } A = 1.$$

Since no other value of x will make a factor vanish, we take any simple values. When $x = 0$, we have

$$3 = A - C, \text{ whence } C = -2.$$

Finally, letting $x = -1$, we have

$$5 = A + 2B - 2C, \text{ whence } B = 0.$$

Therefore $$\frac{x^2 - x + 3}{x^3 - 1} = \frac{1}{x-1} - \frac{2}{x^2 + x + 1}.$$

Ex. 4. $$\frac{2 - 2x + 4x^2}{(1 + x^2)^2(1 - x)} = \frac{Ax + B}{1 + x^2} + \frac{Cx + D}{(1 + x^2)^2} + \frac{E}{1 - x}.$$

The prime factors in the denominators of the first two partial fractions being of the second degree, expressions of the first degree are assumed as numerators.

Clearing of fractions, we have

$$\begin{aligned}
2 - 2x + 4x^2 &= (Ax + B)(1 + x^2)(1 - x) + (Cx + D)(1 - x) + E(1 + x^2)^2 \\
&= (-A + E)x^4 + (A - B)x^3 + (-A + B - C + 2E)x^2 \\
&\qquad + (A - B + C - D)x + (B + D + E).
\end{aligned}$$

Equating coefficients of like powers of x, we obtain

$$-A + E = 0,\ A - B = 0,\ -A + B - C + 2E = 4,$$
$$A - B + C - D = -2,\ B + D + E = 2;$$

whence, $\quad A = 1,\ B = 1,\ C = -2,\ D = 0,\ E = 1.$

Therefore $\dfrac{2 - 2x + 4x^2}{(1 + x^2)^2(1 - x)} = \dfrac{x+1}{1 + x^2} - \dfrac{2x}{(1 + x^2)^2} + \dfrac{1}{1 - x}.$

Ex. 5.

$$\frac{1}{(x + n)(x + n + 1)(x + n + 2)} = \frac{A}{x + n} + \frac{B}{x + n + 1} + \frac{C}{x + n + 2}.$$

Clearing of fractions, we have

$$1 = A(x + n + 1)(x + n + 2) + B(x + n)(x + n + 2)$$
$$+ C(x + n)(x + n + 1).$$

Letting $x = -n$, $\quad$ we have $1 = 2A$, or $A = \frac{1}{2}$;

$x = -n - 1$, $\quad 1 = -B$, or $B = -1$;

$x = -n - 2$, $\quad 1 = 2C$, or $C = \frac{1}{2}$.

Therefore

$$\frac{1}{(x+n)(x+n+1)(x+n+2)} = \frac{1}{2(x+n)} - \frac{1}{x + n + 1} + \frac{1}{2(x+n+2)}.$$

9. The General Term. — The following examples illustrate the method of finding the general term of the expansion of a rational fraction in a series, to ascending powers of x.

Ex. 1. Find the general term of the expansion of $\dfrac{2 + 7x}{1 + x - 2x^2}$.

We have

$$\begin{aligned}
\frac{2 + 7x}{1 + x - 2x^2} &= \frac{3}{1 - x} - \frac{1}{1 + 2x} \\
&= 3(1 + x + x^2 + \cdots + x^n + \cdots) \\
&\quad - [1 + (-2x) + (-2x)^2 + \cdots + (-2x)^n + \cdots].
\end{aligned}$$

The expansions of the above partial fractions, and similar ones, are readily obtained by the formula

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots.$$

The required general term is the sum of the $(n+1)$th terms of the above expansions.

We have $3x^n - (-2x)^n = x^n[3 + (-1)^{n+1}2^n]$.

The expansion of the given fraction can be obtained from this general term. Giving to n the values 0, 1, 2, 3, $\cdots$, we obtain

$$\frac{2+7x}{1+x-2x^2} = 2 + 5x - x^2 + 11x^3 - \cdots + [3 + (-1)^{n+1}2^n]x^n + \cdots.$$

Ex. 2. Find the general term of the expansion of

$$\frac{10-7x+6x^2}{(2-x)(1+x^2)}.$$

We have

$$\begin{aligned}\frac{10-7x+6x^2}{(2-x)(1+x^2)} &= \frac{4}{2-x} + \frac{3-2x}{1+x^2} = \frac{2}{1-\frac{1}{2}x} + \frac{3-2x}{1+x^2}\\ &= 2[1 + \tfrac{1}{2}x + (\tfrac{1}{2}x)^2 + \cdots + (\tfrac{1}{2}x)^{2n} + (\tfrac{1}{2}x)^{2n+1} + \cdots]\\ &\quad + (3-2x)[1 + (-x^2) + (-x^2)^2 + \cdots + (-x^2)^n + \cdots]\\ &= 2[1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \cdots + (\tfrac{1}{2})^{2n}x^{2n} + (\tfrac{1}{2})^{2n+1}x^{2n+1} + \cdots]\\ &\quad + [3 - 3x^2 + 3x^4 - \cdots + (-1)^n 3x^{2n} + \cdots]\\ &\quad + [-2x + 2x^3 - 2x^5 + \cdots + (-1)^{n+1}2x^{2n+1} + \cdots].\end{aligned}$$

Observe that it is necessary to distinguish between even and odd powers of x.

Terms containing *even* powers of x are obtained from

$$(\tfrac{1}{2})^{2n-1}x^{2n} + (-1)^n 3x^{2n}, \; = x^{2n}[(\tfrac{1}{2})^{2n-1} + 3(-1)^n];$$

and terms containing *odd* powers from

$$(\tfrac{1}{2})^{2n}x^{2n+1} + (-1)^{n+1}2x^{2n+1}, \; = x^{2n+1}[(\tfrac{1}{2})^{2n} + 2(-1)^{n+1}].$$

The expansion is readily obtained from these general terms.

EXERCISES III.

Separate the following fractions into partial fractions:

1. $\dfrac{6}{(x-2)(1-2x)}$.

2. $\dfrac{7}{(5+3x)(x+4)}$.

3. $\dfrac{3x-1}{(x+3)(x-2)}$.

4. $\dfrac{1-x}{(3x+2)(x+1)}$.

5. $\dfrac{5}{1-x^2}$.

6. $\dfrac{6x}{x^2-4}$.

7. $\dfrac{1+x}{9-x^2}$.

8. $\dfrac{1}{7x-x^2-12}$.

9. $\dfrac{x^2+2x-1}{9x^2-16}$.

10. $\dfrac{3x+2}{(x^2-1)(x-2)}$.

11. $\dfrac{x^2+90x-9}{6(x^2-9)(x-3)}$.

12. $\dfrac{3x^2+1}{(x+1)(x-1)^2}$.

13. $\dfrac{x^2+5x+10}{(x+1)(x+2)(x+3)}$.

14. $\dfrac{5x(x+3)}{(2x+1)(2x-1)(x+1)}$.

15. $\dfrac{3-x}{(2x+1)(2x+3)(x-1)}$.

16. $\dfrac{x}{(x-1)^3}$.

17. $\dfrac{1}{x^3-1}$.

18. $\dfrac{2}{x^3+1}$.

19. $\dfrac{x+1}{x^3-1}$.

20. $\dfrac{1}{x^4-1}$.

21. $\dfrac{1}{x^2(x^2+1)}$.

22–28. Find the general terms of the expansions, to ascending powers of x, of the fractions in Exx. 5–11.

Find the general term of the expansions of the following fractions, to ascending powers of x:

29. $\dfrac{1}{2x(x^2+1)}$.

30. $\dfrac{5x^2-6x-13}{10(x+3)(1+x^2)}$.

31. $\dfrac{6x+26}{3(x-4)(2+3x^2)}$.

Reversion of Series.

10. If one variable be equal to a series of positive integral ascending powers of a second variable, the second variable can be expressed in a series of positive integral ascending powers of the first. This process is called *reversion of series*.

Ex. 1. Revert the series

$$y = x + 2x^2 + 3x^3 + \cdots.$$

Assume $$x = Ay + By^2 + Cy^3 + \cdots, \tag{1}$$

and substitute in the second member of the last equation the value of y given by the first. Then

$$\begin{aligned} x &= A(x + 2x^2 + 3x^3 + \cdots) + B(x + 2x^2 + 3x^3 + \cdots)^2 \\ &\quad + C(x + 2x^2 + 3x^3 + \cdots)^3 + \cdots \\ &= Ax + \left.\begin{matrix} 2A \\ + B \end{matrix}\right| x^2 + \left.\begin{matrix} 3A \\ +4B \\ + C \end{matrix}\right| x^3 + \cdots. \end{aligned}$$

Hence $$A = 1.$$

$$2A + B = 0, \text{ whence } B = -2;$$
$$3A + 4B + C = 0, \text{ whence } C = 5;$$
etc., etc.

Substituting these values of A, B, C, $\cdots$, in (1), we have

$$x = y - 2y^2 + 5y^3 + \cdots.$$

If the series for y in terms of x contain a term free from x, we must find a value of x in a series of powers of y minus that term.

Ex. 2. Revert the series

$$y = 1 + x + x^2 + x^3 + \cdots,$$

or $$y - 1 = x + x^2 + x^3 + \cdots. \tag{2}$$

Assuming $x = A(y-1) + B(y-1)^2 + \cdots$,

and proceeding as in Ex. 1, we obtain $A = 1$, $B = -1$, $C = 1$.

Therefore $x = (y-1) - (y-1)^2 + (y-1)^3 - \cdots$.

EXERCISES IV.

Revert each of the following series to four terms:

1. $y = x + x^2 + x^3 + \cdots$.
2. $y = x + 3x^2 + 5x^3 + \cdots$.
3. $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$.
4. $y = 1 - x + 2x^2 - \cdots$.
5. $y = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \cdots$.
6. $y = ax + bx^2 + cx^3 + \cdots$.

CHAPTER XXVII.

THE BINOMIAL THEOREM.

1. In Ch. XXII. it was proved by induction that, when n is a positive integer,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{k-1}a^{n-k+1}b^{k-1} + \cdots.$$

We will here give a briefer proof, based upon the theory of combinations.

Consider the following continued product of n factors:

$$n \text{ factors} \begin{cases} a+b \\ a+b \\ \ \cdot\ \cdot\ \cdot \\ \ \cdot\ \cdot\ \cdot \\ a+b \end{cases}$$

The first term of the product is formed by taking an a from each factor, giving a^n. A second term is formed by taking an a from $n-1$ factors and a b from the remaining factor, giving $a^{n-1}b$. But such a term can be formed in as many ways as one b can be taken from n b's, *i.e.*, in ${}_nC_1$ ways. Therefore the product so far is $a^n + {}_nC_1a^{n-1}b$.

A third term is formed by taking an a from $n-2$ factors and a b from the remaining two factors, giving $a^{n-2}b^2$. But such a term can be formed in as many ways as two b's can be taken from n b's, *i.e.*, in ${}_nC_2$ ways. Consequently, the product to this point is $a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2$.

In general, an a can be taken from each of $n-k+1$ factors and a b from each of the remaining $k-1$ factors, giving $a^{n-k+1}b^{k-1}$. But such a term can evidently be formed in ${}_nC_{k-1}$ ways.

We thus obtain

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_{k-1}a^{n-k+1}b^{k-1} + \cdots.$$

But ${}_nC_1 = \binom{n}{1}$, ${}_nC_2 = \binom{n}{2}$, ${}_nC_3 = \binom{n}{3}$, $\cdots$, ${}_nC_{k-1} = \binom{n}{k-1}$.

Therefore, $$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{k-1}a^{n-k+1}b^{k-1} + \cdots.$$

Properties of Binomial Coefficients.

2. The kth term, counting from the beginning of the expansion, contains b^{k-1}, and is ${}_nC_{k-1}a^{n-k+1}b^{k-1}$. The kth term, counting from the end, contains a^{k-1}, and therefore b^{n-k+1}, and is ${}_nC_{n-k+1}a^{k-1}b^{n-k+1}$.

But, by Ch. XXIII., Art. 14, ${}_nC_{k-1} = {}_nC_{n-k+1}$. We therefore conclude:

In the expansion of $(a+b)^n$, wherein n is a positive integer, the coefficients of terms equally distant from the beginning and end of the expansion are equal.

3. By Art. 1, the coefficient of the $(k+1)$th term is ${}_nC_k$. Therefore, by Ch. XXIII., Art. 15, we have:

The greatest binomial coefficient, when n is even, is ${}_nC_{\frac{n}{2}}$; and when n is odd, is ${}_nC_{\frac{n-1}{2}}$, $= {}_nC_{\frac{n+1}{2}}$.

4. In $(1+x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_nx^n$, let $x = 1$.

Then $$2^n = 1 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n.$$

That is, *the sum of the binomial coefficients is 2^n.*

5. From Art. 4, we have

$${}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1.$$

That is, *the total number of combinations of n things, taken one at a time, two at a time, and so on, to n at a time, is $2^n - 1$.*

6. In $(1+x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_nx^n$, let $x = -1$.

Then $$1 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \cdots = 0,$$

or $$1 + {}_nC_2 + {}_nC_4 + \cdots = {}_nC_1 + {}_nC_3 + \cdots.$$

That is, *in the binomial expansion, the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.*

Binomial Theorem for Any Rational Exponent.

7. From Ch. XXII., Art. 4, we have

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots, \tag{1}$$

when n is a positive integer. In this case the expansion ends with the $(n+1)$th term, since the coefficients of the $(n+2)$th and all succeeding terms contain $n-n$, or 0, as a factor. But if n be not a positive integer, the expression on the right of (1) will continue without end, since no factor of the form $n-k+1$ can reduce to 0. Therefore this series will have no meaning unless it be convergent.

8. In Chap. XXV., Art. 27, it was proved that the series

$$1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$

is convergent when x lies between -1 and $+1$. It remains to be proved, therefore, that in this case the above series represents $(1+x)^n$, when n is a fraction or negative.

9. Since the reasoning will turn upon the value of n, we shall call the expression

$$1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$

a function of n, and abbreviate it by $f(n)$, *for all rational values of n.* To understand the following reasoning, the

student should notice that for all positive integral values of n, $(1+x)^n = f(n)$, as, $(1+x)^3 = f(3)$; and that it remains to be proved that $(1+x)^n = f(n)$, when n is a fraction or negative; as, for example, that $(1+x)^{\frac{2}{3}} = f(\frac{2}{3})$.

10. We now have

$$f(m) = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{k-1}x^{k-1} + \cdots$$

$$f(n) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k-1}x^{k-1} + \cdots$$

for all values of x between -1 and $+1$.

We will assume that the product $f(m) \times f(n)$ is a convergent series, when the two series are convergent. The proof of this principle is beyond the scope of this book. We then have

$$\begin{aligned} f(m) \times f(n) = 1 &+ \left[\binom{m}{1} + \binom{n}{1}\right]x + \left[\binom{m}{2} + \binom{m}{1}\binom{n}{1} + \binom{n}{2}\right]x^2 + \cdots \\ &+ \left[\binom{m}{k-1} + \binom{m}{k-2}\binom{n}{1} + \binom{m}{k-3}\binom{n}{2} + \cdots \right. \\ &\left. + \binom{m}{2}\binom{n}{k-3} + \binom{m}{1}\binom{n}{k-2} + \binom{n}{k-1}\right]x^{k-1} + \cdots. \end{aligned}$$

But, by Ch. XXIII., Art. 17,

$$\binom{m}{1} + \binom{n}{1} = \binom{m+n}{1}, \quad \binom{m}{2} + \binom{m}{1}\binom{n}{1} + \binom{n}{2} = \binom{m+n}{2},$$

$$\binom{m}{k-1} + \binom{m}{k-2}\binom{n}{1} + \cdots + \binom{m}{1}\binom{n}{k-2} + \binom{n}{k-1} = \binom{m+n}{k-1};$$

therefore $$f(m) \times f(n) = f(m+n), \tag{1}$$

for all rational values of m and n.

Then $f(m) \times f(n) \times f(p) = f(m+n) \times f(p) = f(m+n+p)$.

In general,

$$f(m) \times f(n) \times f(p) \times \cdots \times f(r) = f(m+n+p+\cdots+r), \tag{2}$$

for all rational values of $m, n, p, \cdots, r$.

11. Fractional Exponents. — Let

$$m = n = p = \cdots = r = \frac{u}{v},$$

wherein u and v are positive integers. Taking v factors, we now have

$$f\left(\frac{u}{v}\right) \times f\left(\frac{u}{v}\right) \times f\left(\frac{u}{v}\right) \times \cdots v \text{ factors} = f\left(\frac{u}{v} + \frac{u}{v} + \frac{u}{v} + \cdots v \text{ summands}\right),$$

or
$$\left[f\left(\frac{u}{v}\right)\right]^v = f\left(\frac{u}{v} \cdot v\right) = f(u).$$

Now, since u is a positive integer, $(1 + x)^u = f(u)$.

Therefore $(1 + x)^u = \left[f\left(\frac{u}{v}\right)\right]^v$, or $(1 + x)^{\frac{u}{v}} = f\left(\frac{u}{v}\right)$.

That is, $$(1 + x)^{\frac{u}{v}} = 1 + \begin{Bmatrix} \frac{u}{v} \\ 1 \end{Bmatrix} x + \begin{Bmatrix} \frac{u}{v} \\ 2 \end{Bmatrix} x^2 + \cdots.$$

12. Negative Exponents, Integral or Fractional. — In (1), Art. 10, let
$$m = -n.$$

We then have $f(-n) \times f(n) = f(n - n) = f(0) = 1,$

since $$f(0) = 1 + 0 \cdot x + \cdots = 1.$$

Therefore $$\frac{1}{f(n)} = f(-n). \qquad (1)$$

Since n is a positive integer or fraction, $(1 + x)^n = f(n)$, and (1) becomes

$$\frac{1}{(1 + x)^n} = f(-n), \text{ or } (1 + x)^{-n} = f(-n).$$

That is, $(1 + x)^{-n} = 1 + \binom{-n}{1} x + \binom{-n}{2} x^2 + \cdots.$

13. Expansion of $(a + b)^n$. — We have

$$(a + b)^n = \left[a\left(1 + \frac{b}{a}\right)\right]^n = a^n\left(1 + \frac{b}{a}\right)^n; \qquad (1)$$

and
$$(a + b)^n = \left[b\left(1 + \frac{a}{b}\right)\right]^n = b^n\left(1 + \frac{a}{b}\right)^n. \qquad (2)$$

When b is numerically less than a,

$$\left(1+\frac{b}{a}\right)^n = 1+\binom{n}{1}\frac{b}{a}+\binom{n}{2}\frac{b^2}{a^2}+\cdots,$$

and, by (1) above,

$$(a+b)^n = a^n\left[1+\binom{n}{1}\frac{b}{a}+\binom{n}{2}\frac{b^2}{a^2}+\cdots\right]$$

$$= a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots. \quad (3)$$

In a similar way it can be shown that, when a is numerically less than b,

$$(a+b)^n = b^n+\binom{n}{1}b^{n-1}a+\binom{n}{2}b^{n-2}a^2+\cdots. \quad (4)$$

Notice that when n is a fraction or negative, formula (3) or (4) must be used according as a is numerically greater or less than b.

14. Ex. Expand $\dfrac{1}{\sqrt[3]{(a-4b^2)}}$ to four terms.

If we assume $a > 4b^2$, we have, by (3), Art. 13,

$$\frac{1}{\sqrt[3]{(a-4b^2)}} = (a-4b^2)^{-\frac{1}{3}} = a^{-\frac{1}{3}}+(-\tfrac{1}{3})a^{-\frac{4}{3}}(-4b^2)$$

$$+\frac{-\frac{1}{3}(-\frac{4}{3})}{1\cdot 2}a^{-\frac{7}{3}}(-4b^2)^2$$

$$+\frac{-\frac{1}{3}(-\frac{4}{3})(-\frac{7}{3})}{1\cdot 2\cdot 3}a^{-\frac{10}{3}}(-4b^2)^3+\cdots$$

$$= \frac{1}{\sqrt[3]{a}}+\frac{4b^2}{3a\sqrt[3]{a}}+\frac{32b^4}{9a^2\sqrt[3]{a}}+\frac{896b^6}{81a^3\sqrt[3]{a}}+\cdots.$$

If $a < 4b^2$, we should have used (4), Art. 13.

Any particular term can be written as in Ch. XXII., Art. 9.

15. Extraction of Roots of Numbers. — Ex. Find $\sqrt{17}$ to four decimal places. We have

$$\begin{aligned}\sqrt{17} &= \sqrt{(16+1)} = 4\left(1+\tfrac{1}{16}\right)^{\frac{1}{2}}\\ &= 4\left[1+\frac{1}{2}\times\frac{1}{16}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot 2}\left(\frac{1}{16}\right)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2\cdot 3}\left(\frac{1}{16}\right)^3+\cdots\right]\\ &= 4\,(1+.03125-.00049+.00002-\cdots)\\ &= 4\times 1.03078 = 4.12312.\end{aligned}$$

Therefore $\sqrt{17} = 4.1231$, to four decimal places.

EXERCISES.

Expand to four terms:

1. $(1+a)^{\frac{1}{2}}$.
2. $(1-x)^{-1}$.
3. $(1-x)^{-3}$.
4. $(1+x^2)^{\frac{3}{2}}$.
5. $(1+x)^{-4}$.
6. $(1-y^2)^{-2}$.
7. $(x^2+y)^{-\frac{2}{3}}$.
8. $(x-y^3)^{-4}$.
9. $(27+5x)^{\frac{2}{3}}$.
10. $(8a^3-3b)^{\frac{1}{3}}$.
11. $(3+2x)^{\frac{3}{4}}$.
12. $(5a^2-3b^3)^{-\frac{2}{3}}$.
13. $\dfrac{1}{\sqrt{(a^2-b^2)}}$.
14. $\dfrac{1}{\sqrt[3]{(a^3-b)}}$.
15. $\dfrac{1}{\sqrt{(2x^{-1}-34^{\frac{1}{2}})^3}}$.

Find the

16. 4th term of $(1-2x)^{\frac{1}{3}}$.
17. 6th term of $(1+a^2b^{-\frac{1}{3}})^{-3}$.
18. 5th term of $(x^{\frac{2}{3}}-x^{-1}y^2)^{-\frac{3}{4}}$.
19. 8th term of $(a^3\sqrt{b}-2b\sqrt[3]{a})^{-\frac{1}{2}}$.
20. k—5th term of $(1+x^{\frac{1}{3}}y^{\frac{1}{2}})^{-2}$.
21. $2k$th term of $[x^2-\sqrt{(xy)}]^{\frac{2}{3}}$.

Find to four places of decimals the values of:

22. $\sqrt{5}$.
23. $\sqrt{27}$.
24. $\sqrt[3]{35}$.
25. $\sqrt[4]{700}$.
26. $\sqrt[5]{258}$.
27. Find the term in $(3x^3-x^2y)^{\frac{5}{3}}$ containing x^2.
28. Find the term in $\left(a+\dfrac{1}{2\sqrt{a}}\right)^{-\frac{1}{2}}$ containing a^{-11}.

CHAPTER XXVIII.

LOGARITHMS.

1. A value of x can always be found to satisfy an equation of the form

$$10^x = n,$$

wherein n is any real positive number. *E.g.*, when $n = 10$, $x = 1$, when $n = 100$, $x = 2$, when $n = 1000$, $x = 3$, etc.

The proof of this principle is beyond the scope of this book.

When n is not an integral power of 10, the value of x is irrational, and can be expressed only approximately. Thus, when $n = 24$, the corresponding value of x has been found to be $1.38021\cdots$, to five decimal places; or

$$10^{1.38021\cdots} = 24.$$

A value of x is called the *logarithm* of the corresponding value of n, and *10* is called the *base*.

In general, a value of x which satisfies the equation $b^x = n$, is called *the logarithm of n to the base b*.

E.g., since $2^3 = 8$, 3 is the logarithm of 8 to the base 2; since $10^2 = 100$, 2 is the logarithm of 100 to the base 10.

The Logarithm of a given number n to a given base b is, therefore, the exponent of the power to which the base b must be raised to produce the number n.

2. The relation $b^x = a$ is also written $x = \log_b a$, read *x is the logarithm of a to the base b.* Thus,

$$2^3 = 8 \quad \text{and } 3 = \log_2 8,$$

$$10^2 = 100 \text{ and } 2 = \log_{10} 100,$$

are equivalent ways of expressing one and the same relation.

3. The theory of logarithms is based upon the idea of representing all positive numbers, in their natural order, as powers of one and the same base.

Thus, 4, 8, 16, 32, 64, etc., can all be expressed as powers of a common base 2; as $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, etc. Since, also, all the numbers intermediate between those given above can be expressed as powers of 2, the exponents of these powers are the logarithms of the corresponding numbers.

The logarithms of all positive numbers to a given base form what is called a **System of Logarithms.** The base is then called the *base of the system.*

It follows from Art. 1, that any positive number except 1 may be taken as the base of a system of logarithms.

EXERCISES I.

Express the following relations in the language of logarithms:

1. $5^2 = 25.$ **2.** $2^5 = 32.$ **3.** $7^3 = 343.$ **4.** $3^7 = 2187.$

Express the following relations in terms of powers:

5. $\log_3 81 = 4.$ **6.** $\log_9 81 = 2.$ **7.** $\log_4 64 = 3.$ **8.** $\log_2 64 = 6.$

Determine the values of the following logarithms:

9. $\log_2 32.$ **10.** $\log_{\frac{1}{2}} 128.$ **11.** $\log_2 .5.$ **12.** $\log_2 .25.$
13. $\log_4 64.$ **14.** $\log_{64} 8.$ **15.** $\log_2 .125.$ **16.** $\log_5 .04.$

To the base 16, what numbers have the following logarithms?

17. $0.$ **18.** $\frac{1}{2}.$ **19.** $-2.$ **20.** $\frac{3}{2}.$ **21.** $-\frac{1}{4}.$

Principles of Logarithms.

4. *The logarithm of* 1 *to any base is* 0. For $b^0 = 1$, or $\log_b 1 = 0$.

5. *The logarithm of the base itself is* 1. For $b^1 = b$, or $\log_b b = 1$.

6. *The logarithm of a product is equal to the sum of the logarithms of its factors;* or,

$$\log_b (m \times n) = \log_b m + \log_b n.$$

Let $\log_b m = x$ and $\log_b n = y$;

then $b^x = m$ and $b^y = n$, and therefore, $mn = b^x b^y = b^{x+y}$.

Translated into the language of logarithms, this result reads

$$\log_b(mn) = x + y.$$

But $x = \log_b m$ and $y = \log_b n$,

and consequently

$$\log_b(mn) = \log_b m + \log_b n,$$

for all positive values of b.

This result may be readily extended to a product of any number of factors. For,

$$\log_b(mnp) = \log_b(mn) + \log_b p = \log_b m + \log_b n + \log_b p.$$

And, in like manner, for any number of factors.

E.g. Given $\log_2 32 = 5$, and $\log_2 64 = 6$; what is the logarithm of 2048 to the base 2?

Since $2048 = 32 \cdot 64$, we have

$$\log_2 2048 = \log_2 32 + \log_2 64 = 5 + 6 = 11$$

7. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor;* or,

$$\log_b (m \div n) = \log_b m - \log_b n.$$

Let $\log_b m = x$ and $\log_b n = y$;

then $b^x = m$ and $b^y = n$, and therefore $m \div n = b^x \div b^y = b^{x-y}$.

In the language of logarithms the last equation is

$$\log_b (m \div n) = x - y = \log_b m - \log_b n,$$

for all positive values of b.

E.g. Given $\log_3 3 = 1$ and $\log_3 2187 = 7$, what is the logarithm of 729 to the base 3?

Since $729 = \frac{2187}{3}$,

we have $\log_3 729 = \log_3 2187 - \log_3 3 = 7 - 1 = 6.$

8. Both m and n may be products, or the quotient of two numbers.

E.g., $$\log_{10}\frac{4\times 5}{9\times 8}=\log_{10}(4\times 5)-\log_{10}(9\times 8)$$
$$=\log_{10}4+\log_{10}5-\log_{10}9-\log_{10}8.$$

9. *The logarithm of the reciprocal of any number is the opposite of the logarithm of the number.*

For, $$\log_b\frac{1}{n}=\log_b 1-\log_b n$$
$$=-\log_b n,\text{ since }\log_b 1=0.$$

E.g., $$\log_2 4=2,\text{ and }\log_2\tfrac{1}{4}=-2.$$

10. *The logarithm of any power, integral or fractional, of a number is equal to the logarithm of the number multiplied by the exponent of the power;* or

$$\log m^p=p\log m.$$

Let $$\log_b m=x,\text{ then }b^x=m.$$

Raising both sides of the last equation to the pth power, we have $b^{px}=m^p$, or $\log_b(m^p)=px=p\log_b m$.

E.g., If $\log_5 25=2$, what is $\log_5 25^3$?

We have $\log_5 25^3=3\log_5 25=3\times 2=6.$

11. When the exponent is a positive fraction whose numerator is 1, this principle may be conveniently stated thus:

The logarithm of a root of a number is the logarithm of the number divided by the index of the root.

For, $$\log_b(m^{\frac{1}{q}})=\frac{1}{q}\log m=\frac{\log_b m}{q}.$$

E.g., If $\log_7 2401=4$, what is $\log_7\sqrt{2401}$?

We have $$\log_7\sqrt{2401}=\frac{1}{2}\log_7 2401=\frac{1}{2}\cdot 4=2.$$

EXERCISES II.

Express the following logarithms in terms of $\log a$, $\log b$, $\log c$, and $\log d$:

1. $\log \frac{abc}{d}$. 2. $\log \frac{d}{abc}$. 3. $\log \frac{ac^2}{bd^2}$. 4. $\log \left(\frac{ac}{bd}\right)^2$.

5. $\log a^{\frac{5}{6}} d^{-\frac{2}{3}} \sqrt{b} \sqrt{c}$. 6. $\log \frac{2\,ab^2}{3\,c\sqrt{d}}$. 7. $\log \frac{a^{-2}b^{\frac{3}{2}}}{\sqrt{(c^5 d^{-3})}}$.

Express the following sums of logarithms as logarithms of products and quotients.

8. $\log a + \log b - \log c$. 9. $\log a - (\log b + \log c)$.

10. $3 \log a - \frac{1}{2} \log (b + c)$. 11. $\frac{1}{2} \log (1 - x) + \frac{3}{2} \log (1 + x)$.

12. $2 \log \frac{a}{b} + 3 \log \frac{b}{a}$. 13. $2 \log a - \frac{2}{3} \log b + \frac{1}{2} \log c$.

Given $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$, $\log_{10} 5 = .69897$, $\log_{10} 7 = .84510$, find the values of the following logarithms, to the base 10:

14. $\log 50$. 15. $\log 6$. 16. $\log 8$. 17. $\log 9$.

18. $\log 12$. 19. $\log 36$. 20. $\log 108$. 21. $\log 4\frac{1}{2}$.

22. $\log 2\frac{2}{3}$. 23. $\log 5\frac{5}{6}$. 24. $\log 5\frac{1}{7}$. 25. $\log 360$.

26. $\log 3072$. 27. $\log 3500$. 28. $\log 5880$.

29. $\log \sqrt{72}$. 30. $\log \sqrt{180}$. 31. $\log \sqrt{1715}$.

32. $\log \frac{\sqrt[5]{490}}{\sqrt[6]{96}}$. 33. $\log \frac{\sqrt[6]{9\frac{3}{5}} \times \sqrt{105}}{\sqrt[3]{72} \times \sqrt[4]{8\frac{2}{5}}}$. 34. $\log \frac{(4\frac{2}{3})^3}{(11\frac{2}{3})^{\frac{3}{2}}}$.

Systems of Logarithms.

12. The two most important systems of logarithms are:

(i.) The system whose base is 10. This system was introduced, in 1615, by the Englishman, Henry Briggs.

Logarithms to the base 10 are called **Common, or Briggs's *Logarithms***.

(ii.) The system whose base is the sum of the following infinite series,

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots.$$

The value of this sum, which to seven places of decimals is 2.7182818, is denoted by the letter e.

Logarithms to the base e are called **Natural Logarithms**; sometimes also **Napierian Logarithms**, in honor of the inventor of logarithms, the Scotch Baron Napier, a contemporary of Briggs. Napier himself did not, however, introduce this system of logarithms.

These two systems are the only ones which have been generally adopted; the common system is used in practical calculations, the natural system in theoretical investigations. The reason that in all practical calculations the common system of logarithms is superior to other systems is because its base 10 is also the base of our decimal system of numeration.

The logarithms of most numbers are irrational, and thus approximate values are used.

Properties of Common Logarithms.

13. In the following articles the subscript denoting the base 10 will be omitted.

We now have

$$(a)\begin{cases} 10^0 = 1, \text{ or } \log 1 = 0; \\ 10^1 = 10, \text{ or } \log 10 = 1; \\ 10^2 = 100, \text{ or } \log 100 = 2; \\ 10^3 = 1000, \text{ or } \log 1000 = 3; \end{cases}$$

$$\cdots\cdots\cdots\cdots$$

$$(b)\begin{cases} 10^{-1} = .1, \text{ or } \log .1 = -1; \\ 10^{-2} = .01, \text{ or } \log .01 = -2; \\ 10^{-3} = .001, \text{ or } \log .001 = -3; \\ 10^{-4} = .0001, \text{ or } \log .0001 = -4; \end{cases}$$

$$\cdots\cdots\cdots\cdots$$

Evidently the logarithms of all positive numbers, except positive and negative integral powers of 10, consist of an integral and a decimal part. Thus, since $10^1 < 85 < 10^2$, we have $1 < \log 85 < 2$, or $\log 85 = 1 +$ *a decimal.*

14. The integral part of a logarithm is called its **Characteristic.**

The decimal part of a logarithm is called its **Mantissa.**

15. Since a number having one digit in its integral part, as 7.3, lies between 10^0 and 10^1, it follows from table (*a*) that its logarithm lies between 0 and 1, *i.e.*, is $0 +$ *a decimal.* Since any number having two digits in its integral part, as 76.4, lies between 10^1 and 10^2, its logarithm lies between 1 and 2, that is, is $1 +$ *a decimal.* In general, since any number having n digits in its integral part lies between 10^{n-1} and 10^n, its logarithm lies between $n - 1$ and n, *i.e.*, is $n - 1 +$ *a decimal.* We therefore have:

(i.) *The characteristic of the logarithm of a number greater than unity is positive, and is one less than the number of digits in its integral part.*

E.g., $\log 2756.3 = 3 +$ *a decimal.*

Since a number less than 1 having no cipher immediately following the decimal point lies between 10^0 and 10^{-1}, it follows from table (*b*) that its logarithm lies between 0 and -1, *i.e.*, is $-1 +$ *a positive decimal.* Since a number less than 1 having one cipher immediately following the decimal point lies between 10^{-1} and 10^{-2}, its logarithm lies between -1 and -2, *i.e.*, is $-2 +$ *a positive decimal.* In general, since a number less than 1 having n ciphers immediately following the decimal point lies between 10^{-n} and $10^{-(n+1)}$, its logarithm lies between $-n$ and $-(n+1)$, *i.e.*, is $-(n+1) +$ *a positive decimal.* We therefore have:

(ii.) *The characteristic of the logarithm of a number less than 1 is negative, and is numerically one greater than the number of ciphers immediately following the decimal point.*

E.g., $\log .00035 = -4 +$ *a positive decimal.*

It follows conversely from (i.) and (ii.):

(iii.) *If the characteristic of a logarithm be $+n$, there are $n+1$ digits in the integral part of the corresponding number.*

(iv.) *If the characteristic of a logarithm be $-n$, there are $n-1$ ciphers immediately following the decimal point of the corresponding number.*

16. It has been found that $538 = 10^{2.73078}$ to five decimal places, or $\log 538 = 2.73078$. We also have

$$\log .0538 = \log \tfrac{538}{10000} = \log 538 - \log 10000 = 2.73078 - 4$$
$$= .73078 - 2;$$
$$\log 5.38 = \log \tfrac{538}{100} = \log 538 - \log 100 = 2.73078 - 2$$
$$= .73078;$$
$$\log 53800 = \log (538 \times 100) = \log 538 + \log 100$$
$$= 2.73078 + 2 = 4.73078.$$

These examples illustrate the following principle:

If two numbers differ only in the position of their decimal points, their logarithms have different characteristics but the same positive mantissa.

17. The characteristic and the mantissa of a number less than 1 may be connected by the decimal point, if the sign $(-)$ be written over the characteristic to indicate that the characteristic only is negative, and not the entire number.

Thus, instead of $\log .00709 = .85065 - 3 = -3 + .85065$, we may write $\bar{3}.85065$; this must be distinguished from the expression -3.85065, in which the integer and the decimal are both negative. Similarly,

$$\log .082 = \bar{2}.91381, \text{ while } \log 820 = 2.91381.$$

Five-Place Table of Logarithms.

18. The logarithms, to the base 10, of a set of consecutive integers have been computed.

In tabulating these logarithms, compactness is important.

For this reason, all unnecessary detail is omitted. Since the characteristic of the logarithm of any number can, as we have seen, be determined by inspection, it is unnecessary to write it with the mantissa in the table. Consequently, only the mantissas, *without the decimal points*, are there given.

Neither is it necessary to give the logarithms of decimal fractions, since their mantissas are the same as the mantissas of the numbers obtained by omitting the decimal point.

The logarithms may be carried to any number of decimal places, and the extent to which they are carried depends upon the degree of accuracy required in their use.

19. The accompanying five-place table gives the mantissas of the logarithms of all consecutive integers from 1 to 9999 inclusive.

In this table the first three figures of each number are given in the column headed N, and the fourth figure in the horizontal line over the table. The first figure, which is the same for all numbers in a given column, is printed in every tenth number only.

The columns headed 0, 1, 2, 3, etc., contain the mantissas, with decimal points omitted.

In the column headed 0, when the first two figures are not printed, they are to be taken from the last mantissa above which is printed in full.

In the columns headed 1, 2, 3, etc., the last three figures only are printed; the first two are to be taken from the column headed 0 in the same horizontal line.

When a star is prefixed to the last three figures of a mantissa, the first two figures are to be taken from the line below.

To Find the Logarithm of a Given Number.

20. When the Number consists of Four or Fewer Figures. — Take the mantissa that is in the horizontal line with the first three figures and in the column under the fourth figure of the given number

Determine the characteristic by Art. 15.

E.g., log 2583 = 3.41212, log 46.32 = 1.66577.

In writing logarithms with negative characteristics it is customary to modify the characteristics so that 10 is uniformly subtracted from the logarithms.

Thus, $\bar{2}.45926 = .45926 - 2 = 8.45926 - 10;$

$\bar{4}.37062 = .37062 - 4 = 6.37062 - 10.$

That is, we add 10 to the negative characteristic, and write -10 after the logarithm.

$$\log .5757 = 9.76020 - 10, \log .02768 = 8.44217 - 10.$$

Observe that the first two figures of the mantissa of log .5757 are taken from the line below, in accordance with the directions in Art. 19.

If the given number consists of fewer than four figures, annex ciphers until it has four figures, in taking the mantissa from the table.

E.g., mantissa of log 78 = mantissa of log 7800 = .89209,

and log 78 = 1.89209.

In like manner,

$$\log 583 = 2.76567, \log .02 = 8.30103 - 10.$$

21. When the Number consists of more than Four Significant Figures. — The method used is called *interpolation*, and depends upon the following property of logarithms:

The difference between two logarithms is very nearly proportional to the difference between the corresponding numbers when this difference is small.

The error made by assuming that these differences are exactly proportional will be so small that it may be neglected.

Ex. 1. Find log 27845.

Omitting, for the moment, the decimal points from the mantissas, we have

mantissa of log 27850 = 44483,

mantissa of log 27840 = 44467,

difference of mantissas = 16.

Let x stand for the difference between the mantissas of log 27845 and log 27840; that is, for the *correction* to be added to the smaller mantissa to give the required mantissa.

Then, by the above property,

$$\frac{x}{16}=\frac{27845-27840}{27850-27840}=\frac{5}{10}=.5.$$

Whence $$x = .5 \times 16 = 8.$$

Therefore, mantissa of $\log 27845 = 44467 + 8 = 44475$,

and $$\log 27845 = 4.44475.$$

Observe that, by Art. 16, the mantissa of log 27850 is the same as the mantissa of log 2785. In subsequent work such ciphers will be omitted.

The method can now be stated more concisely for practical work:

Subtract the mantissa corresponding to the first four figures of the given number from the next mantissa in the table; multiply this difference by the remaining figure or figures of the given number, treated as a decimal; add the product to the first (and smaller) mantissa.

Prefix finally the proper characteristic.

In thus finding the mantissa, a decimal point in the given number is ignored, in accordance with Art. 16.

The difference between two consecutive mantissas in the table is called the **Tabular Difference.**

Ex. 2. Find log 78.1283.

We have
$$\begin{aligned}\text{mantissa of } \log 7813 &= 89282,\\ \text{mantissa of } \log 7812 &= 89276,\\ \text{tabular difference} &= \quad\;\; 6,\\ \text{correction} &= .83 \times 6 = 4.98,\\ \text{mantissa of } \log 781283 &= 89276 + 5 = 89281.\end{aligned}$$

Therefore $$\log 78.1283 = 1.89281$$

Observe that the correction added to the mantissa of log 7812 is 5, the nearest integer to 4.98.

22. In the table of logarithms a column containing the required corrections (head **Pp. Pts.**, *i.e.*, proportional parts) is given. In this column there are several small tables, each containing two columns of numbers. One of these columns consists of the consecutive numbers 1 to 9; the other, headed by a tabular difference, contains the correction corresponding to each one of the figures 1 to 9, when it is the *fifth* figure of the number whose logarithm is required. When it is the *sixth* figure, the corresponding tabular correction must evidently be divided by 10; when it is the *seventh* figure, by 100; and so on.

Thus, in Ex. 1 of the preceding article, we take the correction opposite 5, under the tabular difference 16, and obtain 8, as before.

In Ex. 2, we take the following corrections from the column headed by the tabular difference 6:

$$\begin{array}{ll} \text{for } 8, & \text{correction} = 4.8 \\ \text{for } 3, & \text{correction} = \underline{0.18} \\ & \text{final correction} = 4.98, \text{ as before.} \end{array}$$

Observe that the correction for the sixth figure of the given number does not affect the result.

Ex. 3. Find the log .0128546.

$$\begin{array}{lr} \text{We have} & \text{mantissa of } \log 1286 = 10924, \\ & \text{mantissa of } \log 1285 = 10890, \\ & \text{tabular difference} = \quad 34. \end{array}$$

From the column of proportional parts headed by 34, we obtain:

$$\begin{aligned} \text{correction for fifth figure } 4 &= 13.6 \\ \text{correction for sixth figure } 6 &= \underline{\ \ 2.04} \\ \text{total correction} &= 15.64 \end{aligned}$$

Therefore, mantissa of log 128546 = 10890 + 16 = 10906,

and $\log .0128546 = 8.10906 - 10.$

Observe that in this example the correction for the sixth figure does affect the result.

EXERCISES III.

Verify the following statements:

1. $\log 13 = 1.11394$.
2. $\log 14.84 = 1.17143$.
3. $\log 73000 = 4.86332$.
4. $\log 5884.4 = 3.76970$
5. $\log .031586 = 8.49949 - 10$.
6. $\log .00391857 = 7.59313 - 10$.

Find the logarithms of each of the following numbers:

7. 5.	8. 18.	9. 540.	10. 3876.
11. 2076.	12. 59.80.	13. 1.87.	14. .01832.
15. .0004129.	16. 63072.	17. 59.836.	18. 4376.4.
19. .070518.	20. 185462.		21. .00103987.

To find a Number from its Logarithm.

23. Mantissa given in the Table. — If the mantissa of the given logarithm is found in the table, the first three figures of the required number will be in the same line with it in the column headed N, and the fourth figure over the column in which the given mantissa stands.

The characteristic is determined by Art. 15 (iii.) and (iv.).

Ex. 1. Find the number whose logarithm is 4.82099. The mantissa .82099 corresponds to the number 6622; but since the given characteristic is 4, the required number must have five integral places.

Consequently $\qquad 4.82099 = \log 66220.$

Ex. 2. Find the number whose logarithm is $8.78625 - 10$. The mantissa .78625 corresponds to the number 6113; but since the characteristic is -2, the required number must be a decimal having its first significant figure in the second decimal place.

Consequently $\qquad 8.78625 - 10 = \log .06113.$

24. Mantissa not given in the Table. — The method employed is the converse of that used in Art. 21 to find the logarithms of numbers that consist of more than four significant figures.

Ex. 1. Find the number whose logarithm is 2.81727.

We have

given mantissa = 81727;

next smaller mantissa = 81723, corresponding number = 6565;

next larger mantissa = 81730, corresponding number = 6566.

Let x stand for the difference between 6565 and the required number; that is, for the correction to be added to 6565.

We then have

$$\frac{x}{6566-6565}=\frac{81727-81723}{81730-81723}, \text{ or } \frac{x}{1}=\frac{4}{7}=.6,$$

corrected for the first decimal place. Notice that the significance of the decimal point in the result is that the correction is to be *annexed as an additional figure* to the smaller number.

Therefore, the figures in the required number are 65656; and since the characteristic of the given logarithm is 2, there are only three integral places. Hence 2.81727 = log 656.56.

This process may also be stated concisely for practical work:

Take the mantissa next smaller and the mantissa next larger than the given mantissa, and note the numbers corresponding; next divide the difference between the given mantissa and the next smaller by the difference between the next larger and the next smaller. Annex the quotient to the number corresponding to the smaller mantissa, neglecting the decimal point of the quotient.

Place the decimal point in the number thus obtained as it is determined by the given characteristic.

Ex. 2. Find the number whose logarithm is 7.18281 − 10.

We have

given mantissa = 18281;

next smaller mantissa = 18270, corresponding number = 1523;

next larger mantissa = 18298, corresponding number = 1524.

Hence the correction to be annexed to 1523 is

$$\frac{18281-18270}{18298-18270}, =\frac{11}{28}, =.39+$$

Therefore the figures of the required number are 152339; and since the characteristic of the given logarithm is -3, there must be two ciphers between the decimal point and the first significant figure.

Consequently $7.18281 - 10 = \log .00152339$.

In general, in using a five-place table, the numbers corresponding to given mantissas should be carried to only *five* significant figures, as in Ex. 1.

But with mantissas in the first two pages of the table, the corresponding numbers may be carried to six figures. The reason being that the tabular differences later become so small that the correction for a sixth figure will not in general affect the result. See Exx. 2–3, Art. 22.

25. The correction to be added to the number corresponding to the next smaller mantissa may also be taken from the column of proportional parts.

In this column turn to the table headed by the number which is equal to the difference between the next larger and the next smaller mantissa. As the first figure of the correction take the figure in this table which is opposite the proportional part nearest to the difference between the given mantissa and the next smaller mantissa.

If a second figure in the correction is to be found, we should take as the first figure that figure which is opposite the proportional part *next smaller* than the difference between the given mantissa and the next smaller.

Multiply by 10 the difference between the proportional part already used and the difference between the given mantissa and the next smaller, and take the product as a proportional part in determining the second figure of the correction; and so on.

Thus, in Ex. 1 of the preceding article, we turn to the column headed by the tabular difference 7. The proportional part in this table that is nearest to 4 (the difference between the given mantissa and the next smaller) is 4.2; the number *opposite* 4.2 is 6, the correction previously obtained.

In Ex. 2, we turn to the column headed by the tabular difference 28. The proportional part *next smaller* than 11 (the difference between the given mantissa and the next smaller) is 8.4; the figure opposite 8.4 is 3, the first figure of the correction.

We next multiply 2.6 $(= 11 - 8.4)$ by 10, and take the product 26 as a proportional part. The figure opposite 25.2 (nearest to 26) in the column headed by 28 is 9, the second figure of the correction. Therefore, the required correction is found to be 39, as before.

EXERCISES IV.

Verify the following statements:

1. $\log x = 3.14926,\quad x = 1410.13.$
2. $\log x = 1.59187,\quad x = 39.073.$
3. $\log x = .34159,\quad x = 2.1958.$
4. $\log x = 9.57187 - 10,\quad x = .37314.$
5. $\log x = 7.83957 - 10,\quad x = .0069115.$
6. $\log x = 6.18953 - 10,\quad x = .00015471.$

Find the numbers whose logarithms are:

7. 2.26150.	**8.** .59726.	**9.** $8.94655 - 10$.
10. 3.88825.	**11.** 6.19815.	**12.** $6.72576 - 10$.
13. 4.98880.	**14.** 1.68417.	**15.** $9.23360 - 10$.

Cologarithms.

26. The **Cologarithm** of a number, or, as it is sometimes called, the *Arithmetical Complement* of the logarithm, is defined as the logarithm of the reciprocal of the number.

That is, $\operatorname{colog} n = \log \frac{1}{n} = \log 1 - \log n = 0 - \log n.$

We thus see that the cologarithm of a number is obtained by subtracting its logarithm from 0. But this step would leave the mantissa as well as the characteristic negative. To avoid a negative mantissa, therefore, we subtract the logarithm from $10 - 10, = 0$.

Ex. 1. Find the colog 3.

Subtracting $\log 3, = .47712$, from $10 - 10$, we have

$$\begin{array}{r} 10. \qquad -10 \\ .47712 \qquad\; \\ \hline 9.52288 - 10 \end{array}$$

Therefore $\operatorname{colog} 3 = 9.52288 - 10.$

Ex. 2. Find colog .0054.

Subtracting $\log .0054, = 7.73239 - 10$, from $10 - 10$, we have

$$\begin{array}{r} 10. \qquad -10 \\ 7.73239 - 10 \\ \hline 2.26761 \qquad\; \end{array}$$

Therefore $\operatorname{colog} .0054 = 2.26761.$

EXERCISES V.

Verify the following statements:

1. $\operatorname{colog} 543 = 7.26520 - 10.$
2. $\operatorname{colog} 72.318 = 8.14075 - 10.$
3. $\operatorname{colog} 8.9134 = 9.04996 - 10.$
4. $\operatorname{colog} .38145 = .41856.$
5. $\operatorname{colog} .051984 = 1.28413.$
6. $\operatorname{colog} .0091437 = 2.03887.$

Find the cologarithm of each of the following numbers:

7. 5817. 8. .6305. 9. .009812. 10. 763.85.

11. 15.482. 12. 7.00386. 13. .000594. 14. 32581.9

Applications.

27. Ex. 1. Compute the value of x, when

$$x = 53.847 \times .0085965.$$

$$\log x = \log 53.847 + \log .0085965.$$

$$\begin{aligned} \log 53.847 &= 1.73117 \\ \log .0085965 &= \underline{7.93433 - 10} \\ \log x &= 9.66550 - 10 \\ x &= .46291. \end{aligned}$$

Ex. 2. Compute the value of x, when

$$x = 8.4394 \div .31416.$$

$$\log x = \log 8.4394 + \operatorname{colog} .31416.$$

$$\begin{aligned} \log 8.4394 &= .92631 \\ \operatorname{colog} .31416 &= \underline{.50285} \\ \log x &= 1.42916 \\ x &= 26.863. \end{aligned}$$

Ex. 3. Compute the value of x, when

$$x = \frac{6.4319 \times .59218}{7.9254 \times .062547}.$$

$$\log x = \log 6.4319 + \log .59218 + \operatorname{colog} 7.9254 + \operatorname{colog} .062547.$$

$$\begin{aligned} \log 6.4319 &= .80834 \\ \log .59218 &= 9.77246 - 10 \\ \operatorname{colog} 7.9254 &= 9.10098 - 10 \\ \operatorname{colog} .062547 &= \underline{1.20379 } \\ \log x &= 20.88557 - 20 \\ &= .88557. \\ x &= 7.6837. \end{aligned}$$

Ex. 4. Find the value of x, when

$$x = .5318^4.$$

$$\begin{aligned} \log x &= 4 \log .5318 \\ &= 4(9.72575 - 10) \\ &= 38.90300 - 40 \\ &= 8.90300 - 10. \\ x &= .079983. \end{aligned}$$

Ex. 5. Find the value of $\sqrt[3]{-.031459}$.

Since a negative number cannot be expressed as a power of $+10$, such a number does not have a logarithm. In this example, therefore, and in all similar examples, we first determine the sign of the result. We then find the value of the expression obtained by changing each sign $-$ to $+$, and to that result prefix the sign previously determined.

The sign of the result of this example is $-$

Let $$x = \sqrt[3]{.031459}.$$

Then
$$\begin{aligned}\log x &= \tfrac{1}{3}\log .031459\\ &= \tfrac{1}{3}(28.49775 - 30)\\ &= 9.49925 - 10,\end{aligned}$$

and $$x = .31568.$$

Therefore, the required result is $-.31568$.

Observe that in dividing log .031459 by 3, we first modified the characteristic so that the number, 30, which is subtracted from the logarithm is 10 times the divisor; that is, so that the quotient obtained by dividing this number by 3 is 10.

Ex. 6. Compute the value of x, when

$$x = \frac{4.5921 \times \sqrt[3]{.021946}}{.059318 \times .41587^3}.$$

$$\log x = \log 4.5921 + \tfrac{1}{3}\log .021946 + \operatorname{colog} .059318 \\ + 3 \operatorname{colog} .41587.$$

$$\begin{aligned}\log 4.5921 &= .66201\\ \tfrac{1}{3}\log .021946 = \tfrac{1}{3}(28.34135 - 30) &= 9.44712 - 10\\ \operatorname{colog} .059318 &= 1.22681\\ 3\operatorname{colog} .41587 = 3 \times .38104 &= \underline{1.14312}\\ \log x &= 12.47906 - 10\\ &= 2.47906.\\ x &= 301.34.\end{aligned}$$

Ex. 7. Compute the value of x, when

$$x = \sqrt[3]{\frac{5.4318 \times \sqrt{.31459}}{7.1938 \times .2934^2}}.$$

For convenience in arranging the logarithmic work, we first cube both members of this equation, and obtain

$$x^3 = \frac{5.4318 \times \sqrt{.31459}}{7.1938 \times .2934^2}. \qquad (1)$$

Taking logarithms, we have

$$3 \log x = \log 5.4318 + \tfrac{1}{2} \log .31459 + \operatorname{colog} 7.1938 + 2 \operatorname{colog} .2934. \quad (2)$$

In practice, step (1) should be performed mentally, and the result (2) be at once written.

$$\begin{aligned}
\log 5.4318 &= .73494 \\
\tfrac{1}{2} \log .31459 = \tfrac{1}{2}(19.49775 - 20) &= 9.74887 - 10 \\
\operatorname{colog} 7.1938 &= 9.14304 - 10 \\
2 \operatorname{colog} .2934 = 2 \times 0.53254 &= 1.06508 \\
\hline
3 \log x &= 20.69193 - 20 \\
&= .69193. \\
\log x &= .23064. \\
x &= 1.70076.
\end{aligned}$$

EXERCISES VI.

Find the values of each of the following expressions:

1. 31.834×185.592. **2.** $8.0043 \times .5319$.

3. $.004893 \times 6.5942$. **4.** $(-.0514) \times .123857$.

5. $\dfrac{.78}{347}$. **6.** $\dfrac{1539}{78395}$. **7.** $\dfrac{19.7939}{3892.7}$.

8. $\dfrac{380.14 \times (-.0576)}{7.3792}$. **9.** $\dfrac{(-9.7408) \times .000395}{36.937}$.

10. $\dfrac{5.83 \times 91.358}{.00479}$. **11.** $\dfrac{57.13 \times 9.0047}{5.382 \times .07235}$.

12. $\dfrac{4.9 \times (-306) \times 48.3}{100.088 \times 2.9 \times .081}$. **13.** $\dfrac{.79 \times 891.3 \times .00099}{(-10.236) \times .07 \times .0031}$.

14. 7.0435^3. **15.** $.31844^4$. **16.** 2.3817^5.

17. $(3.68 \times .97)^4$. **18.** $(.7918 \times 3.17)^5$.

19. $[.034 \times (-4.9738)]^4$. **20.** $(17.19 \times .00001986)^5$.

21. $\sqrt[5]{13}$.
22. $\sqrt[7]{-251}$.
23. $\sqrt[8]{39.837}$.
24. $\sqrt[10]{163.4^3}$.
25. $\sqrt[5]{.31492^2}$.
26. $\sqrt[6]{1.0031}$.
27. $\sqrt[3]{\frac{14}{15}}$.
28. $\sqrt[4]{\frac{11}{17}}$.
29. $\sqrt[5]{\frac{21}{314}}$.
30. $\frac{2}{3}\sqrt[3]{\frac{5}{6}}$.
31. $2\frac{1}{2}\sqrt[4]{\frac{7}{8}}$.
32. $7\frac{3}{4}\sqrt[6]{\frac{2}{3}}$.
33. $(.74\sqrt[3]{8.21})^4$.
34. $(5.21\sqrt[5]{.3817})^6$.
35. $\frac{3}{4}\sqrt[3]{-5} \times \sqrt[4]{17}$.
36. $3\frac{5}{6}\sqrt[4]{.38} \times \sqrt[5]{7.3815}$.
37. $\sqrt[3]{(.25\sqrt{3})}$.
38. $\sqrt[5]{(112.34\sqrt[3]{.003914})}$.
39. $\sqrt[8]{(17.2\sqrt[3]{.718})}$.
40. $\sqrt[11]{(-23\sqrt[7]{.18943})}$.
41. $5.341\sqrt[4]{(27.39\sqrt[3]{.1439})}$.
42. $23.491^2\sqrt[5]{(.18\sqrt[4]{17.3})}$.
43. $\sqrt[6]{\dfrac{3.19\sqrt[5]{-9.2614}}{.519^2\sqrt{117.38}}}$.
44. $5.14\sqrt[7]{\dfrac{.1934\sqrt[3]{.13945}}{583.5\sqrt{27.3}}}$.

Exponential Equations.

28. An Exponential Equation is an equation in which the unknown number appears as an exponent of a known or an unknown number, as $a^x = b$.

Ex. 1. Solve the equation $3^x = 9$.

Taking logarithms, $x \log 3 = \log 9 = 2 \log 3$.

Hence $x = 2$.

This result could have been obtained by inspection, by writing the given equation $3^x = 3^2$.

Ex. 2. Find the value of x in $3^x = 5$.

$$3^x = 5;$$

taking logarithms, $x \log 3 = \log 5$;

whence $$x = \frac{\log 5}{\log 3} = \frac{.69897}{.47712} = 1.46497.$$

Ex. 3. Find the value of x in the following equation

$$2^{3x+1} = 7^{2x-1};$$

taking logarithms, $(3x+1)\log 2 = (2x-1)\log 7.$

Removing parenthesis, $3x\log 2 + \log 2 = 2x\log 7 - \log 7,$

or $$x(3\log 2 - 2\log 7) = -\log 7 - \log 2;$$

whence $$x = \frac{\log 7 + \log 2}{2\log 7 - 3\log 2}$$

$$= \frac{.84510 + .30103}{1.69020 - .90309}$$

$$= \frac{1.14613}{.78711} = 1.4561.$$

EXERCISES VII.

Solve the following exponential equations:

1. $2^x = 64.$ **2.** $3^x = 81.$ **3.** $2^{x-1} = .5^{2x-5}.$

4. $(-8)^{-x} = 16.$ **5.** $4^{3x-1} = .5^{x-5}.$ **6.** $4^x = 8.$

7. $8^x = 32.$ **8.** $5^x = (\sqrt{5})^{-1}.$ **9.** $4^{x+1} = 8\cdot 2^{x+2}.$

10. $25^{3x-1} = 625\cdot 5^{x+3}.$ **11.** $7^{\sqrt{(x-3)}} = 343^{-1}\cdot 49^{\sqrt{(x-3)}}.$

12. $27^{\sqrt{(x-3)}} = (\sqrt{3})^{2\sqrt{(x+3)}}.$ **13.** $\sqrt{a^{11-x}} = a^{8-x}.$

14. $\sqrt[3]{a^{x+2}} = \sqrt{a^{x-3}}.$ **15.** $\sqrt{a^{3-4x}} \div \sqrt[5]{a^{6-7x}} \times a^{\frac{9}{2}} = 1.$

16. $(\frac{1}{5})^x = 25.$ **17.** $(\frac{1}{2})^{x-7} = 64.$

18. $(\frac{27}{19})^{11x-5} = (\frac{19}{27})^{7x-3}.$ **19.** $(\frac{4}{3})^{4x-7} = .75^{2-3x}.$

20. $4^x - 6\cdot 2^x + 8 = 0.$ **21.** $9^x + 243 = 36\cdot 3^x.$

22. $3^{\log x} = 9.$ **23.** $5^{\log 2x} = 625.$ **24.** $16^{\log 3x} = 32^{\log x}.$

25 $5^x = 10.$ **26.** $16^x = 45.$ **27.** $11^x = 310.$

28. $25^x = 10.$ **29.** $7^x = 300.$ **30.** $3.594^x = 359600.$

31. $\sqrt[x]{9.8926} = 1.29.$ **32.** $5^x = 7^{3.14}.$ **33.** $x^{\sqrt{2}} = \sqrt[3]{3}.$

34. $5^{x+3} = 1000.$ **35.** $7^{x+1} = 5.$ **36.** $1.58^{x-5} = 9.847.$

37. $5^{x+1} = 11^{x-1}.$ **38.** $3^{x+7} = 7^{x+3}.$

39. $31^{x+3} = 25^{x+4}.$ **40.** $35^{x+2} = 40^{x-1}.$

Logarithmic Equations.

29. Ex. 1. Solve the equation $\frac{1}{2}\log(x-9)+\log\sqrt{(2x-1)}=1$.

By the principles of logarithms, we obtain successively

$$\log\sqrt{(x-9)}+\log\sqrt{(2x-1)}=\log 10,$$
$$\log\sqrt{[(x-9)(2x-1)]}=\log 10.$$

Therefore $$\sqrt{[(x-9)(2x-1)]}=10,$$

or $$2x^2-19x+9=100.$$

The roots of this equation are 13 and $-\frac{7}{2}$.

Ex. 2. Solve the equation

$$\log(x+12)-\log x=0.8451+\log(6-6x).$$

By the principles of logarithms,

$$\log\frac{x+12}{x}=\log 7(6-5x), \text{ since } 0.8451=\log 7.$$

Consequently $\dfrac{x+12}{x}=42-35x$, or $x+12=42x-35x^2$.

The roots of this equation are $\frac{4}{7}$ and $\frac{21}{35}$.

Ex. 3. Solve the equation $x^{\log x}=100x$.

Taking logarithms, we obtain

$$(\log x)^2=\log 100+\log x, \text{ or } (\log x)^2-\log x=2.$$

Solving this equation as a quadratic in $\log x$, we obtain

$$\log x=2, \text{ or } x=100;\ \log x=-1, \text{ or } x=\tfrac{1}{10}.$$

EXERCISES IV.

Solve the following logarithmic equations:

1. $\log x+\log(x+3)=1$.
2. $\log 4+2\log x=2$.
3. $\log 8+3\log x=3$.
4. $2\log x=1+\log(x+\frac{11}{10})$.
5. $\log\sqrt{(7x+5)}+\log\sqrt{(2x+3)}=1+\log\frac{9}{2}$.
6. $\log(7-9x)^2+\log(3x-4)^2=2$.
7. $\log(x+\sqrt{x})+\log(x-\sqrt{x})=\log 4+\log x^2-\log x$.
8. $\dfrac{\log x^2}{\log(3x-16)}=2$.
9. $\dfrac{\log(2x-3)}{\log(4x^2-15)}=\frac{1}{2}$.
10. $\dfrac{\log(35-x^3)}{\log(5-x)}=3$.

Compound Interest and Annuities.

30. To find the compound interest, I, and the amount, A, of a given principal, P, for n years at r per cent.

If the interest is payable annually, the amount of \$1 at the end of one year will be $1 + r$ dollars, and the amount of P dollars will be $P(1 + r)$ dollars. This amount, $P(1 + r)$, becomes the principal at the beginning the second year. Therefore, at the end of the second year the amount will be $P(1 + r) \times (1 + r), = P(1 + r)^2$ dollars, and so on.

Therefore, at the end of n years the amount will be $P(1 + r)^n$ dollars, or

$$A = P(1 + r)^n.$$

31. This formula can be used not only to find A, but also to find P, r, or n, when the three other quantities are given. Thus,

$$P = \frac{A}{(1 + r)^n}.$$

32. An **Annuity** is a fixed sum of money, payable yearly, or at other fixed intervals, as half-yearly, once in two years, etc.

33. To find the present value, P, of an annuity of A dollars, payable yearly for n years, at r per cent.

The present worth of the first payment is $\frac{A}{1 + r}$ dollars, of the second payment is $\frac{A}{(1 + r)^2}$ dollars, and, in general, of the nth payment is $\frac{A}{(1+r)^n}$ dollars.

Therefore the present worth of all the payments is

$$\frac{A}{1 + r} + \frac{A}{(1 + r)^2} + \cdots + \frac{A}{(1 + r)^n} = \frac{\frac{A}{1 + r}\left[1 - \left(\frac{1}{1 + r}\right)^n\right]}{1 - \frac{1}{1 + r}}.$$

Multiplying numerator and denominator by $1 + r$, we have

$$P = \frac{A}{r}\left[1 - \frac{1}{(1 + r)^n}\right].$$

Ex. 1. Find the amount of $500 for 8 years at 5% compound interest.

$$A = P(1+r)^n = 500 \times 1.05^8.$$

$$\log A = \log 500 + 8 \log 1.05.$$

$$\begin{aligned} \log 500 &= 2.69897 \\ 8 \log 1.05 &= \underline{.16952} \\ \log A &= 2.86849 \\ A &= 738.73. \end{aligned}$$

Therefore the required amount is $ 738.73.

Ex. 2. Find the present value of an annuity of $1000 for 6 years, if the current rate of interest is 5%.

$$P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right] = \frac{1000}{.05}\left[1 - \frac{1}{1.05^6}\right].$$

We will first compute 1.05^6,

$$\begin{aligned} \log (1.05)^6 &= 6. \log 1.05 \\ &= 6 \times .02119 \\ &= .12714. \\ (1.05)^6 &= 1.34012. \end{aligned}$$

We then have

$$P = \frac{1000}{.05}\left[1 - \frac{1}{1.34012}\right] = 20000 \times \frac{.34012}{1.34012},$$

$$\log P = \log 20000 + \log .34012 + \text{colog } 1.34012.$$

$$\begin{aligned} \log 20000 &= 4.30103 \\ \log .34012 &= 9.53163 - 10 \\ \text{colog } 1.34012 &= \underline{9.87286 - 10} \\ \log P &= 23.70552 - 20 \\ &= 3.70552. \\ P &= 5076. \end{aligned}$$

Therefore the present value of the annuity is $ 5076.

EXERCISES IX.

Find the amount at compound interest:

1. Of $3600 for 5 years at $4\frac{1}{2}\%$.

2. Of $1875.50 for 8 years at 5%.

3. Of $12,350 for 6 years at $3\frac{1}{2}\%$.

4. Of $21,580 for 7 years 4 months at 4%.

Find the principal that will amount to:

5. $7913 in 5 years at 5% compound interest.

6. $14,770 in 10 years at $4\frac{1}{2}\%$ compound interest.

7. $11,290 in 8 years at 4% compound interest.

8. $11,090 in 6 years 6 months at 3% compound interest.

9. In what time, at 4%, will $8010 amount to $11,400 at compound interest?

10. In what time, at $4\frac{1}{2}\%$, will $3530 amount to $5987, if the interest is compounded semi-annually?

Find the rate of compound interest:

11. If $1110 amounts to $1640 in 8 years.

12. If $3750 amounts to $6070 in 14 years.

Find the present value of an annuity:

13. Of $1000 for 10 years, if the current rate of interest is 4%.

14. Of $1250 for 8 years, if the current rate of interest is $4\frac{1}{2}\%$.

15. Of $2500 for 10 years, if the current rate of interest is 5%.

16. Of $3000 for 12 years, if the current rate of interest is 6%.

CHAPTER XXIX.

PROBABILITY.

1. In this chapter we shall consider the likelihood that an event, about whose happening there is uncertainty, will happen, or fail to happen.

Thus, if a coin be tossed once, it may fall heads up, but it is not certain to so fall. It may fall tails up. One way of falling is as likely to happen as the other. Now, $\frac{1}{2}$ of the whole number of ways in which a coin can fall is heads up. It seems natural, therefore, to take $\frac{1}{2}$ as the mathematical expression of the likelihood, or probability, that the coin will fall heads up. Then, $\frac{1}{2}$ is also the probability that the coin will fall tails up.

Again, let 4 white balls and 6 red balls be placed in a box, and one ball be drawn at random. If the balls cannot be distinguished by the sense of touch, one ball is as likely to be drawn as any other. Now, one ball can be drawn in 10 different cases, in 4 of which a white ball can be drawn. That is, the number of cases in which a white ball can be drawn is $\frac{4}{10}$, $= \frac{2}{5}$, of the whole number of cases. We therefore take $\frac{2}{5}$ as the mathematical expression of the probability of drawing at random a white ball. The probability of not drawing a white ball, which is the same as the probability of drawing a red ball, is evidently $\frac{3}{5}$.

If data relating to the number of times an event has happened in a large number of cases be collected, these data will indicate quite surely how often the same event will happen in the same number of cases under similar conditions.

Thus, from tables used by life insurance companies, we find that of 95,965 healthy persons of sixteen, 95,293 have lived to

be seventeen. We therefore take $\frac{95293}{95965}$ as the probability that a person of sixteen, in good health, will live to be seventeen.

2. The considerations of the preceding article naturally lead to the following definitions:

The **Favorable Cases** are those in which an event can happen, or has happened in an extended number of cases.

The **Unfavorable Cases** are those in which the event can fail to happen, or has failed to happen in an extended number of cases.

The **Probability** that an event will happen is the ratio of the number of favorable cases to the whole number of cases.

Evidently the probability that an event will not happen is the ratio of the number of unfavorable cases to the whole number of cases.

If a be the number of cases in which an event can happen, and b be the number of cases in which it can fail to happen, and each case be equally likely to happen, we have:

$\dfrac{a}{a+b}$ *is the probability that the event will happen;*

$\dfrac{b}{a+b}$ *is the probability that the event will not happen.*

The **Odds** in favor of an event is defined as the ratio of the number of favorable cases to the number of unfavorable cases.

That is, $\dfrac{a}{b}$ *are the odds in favor of the event;*

in like manner, $\dfrac{b}{a}$ *are the odds against the event.*

3. Since an event is certain to happen or fail to happen, the number of ways favorable to its *happening-or-failing* is $a+b$. Therefore, the probability of the event's happening-or-failing, that is, *certainty*, is

$$\frac{a+b}{a+b}=\frac{a}{a+b}+\frac{b}{a+b}=1.$$

4. If P be the probability that an event will happen, it follows from the preceding article that $1-P$ is the probability that the event will not happen.

Ex. What is the probability of throwing at least 4 in a single throw with two dice?

The number of cases favorable to throwing at least 4 is the number of cases in which 4, 5, 6, ⋯, 12 can be thrown.

The number of unfavorable cases is the number of cases in which 2 and 3 can be thrown.

The required probability can be obtained most readily by first finding the probability of the event's not happening.

The sum 2 can be thrown in one case, 1, 1. The sum 3 can be thrown in two cases, 1, 2 and 2, 1. The two dice can be thrown in $6 \times 6, = 36$, different cases, counting 4, 5 and 5, 4, say, as different throws.

Therefore, the probability of *not* throwing a sum at least 4 is $\frac{3}{36}, = \frac{1}{12}$; and hence, the required probability is $1 - \frac{1}{12}, = \frac{11}{12}$.

5. Ex. A father of thirty-five has a son of twelve. What is the probability that both will be alive thirty years hence?

From the table of mortality given below, we find that of 82,581 persons of thirty-five, 46,754 live to be sixty-five; that of 98,650 persons of twelve, 77,012 live to be forty-two. Now, each of the 46,754 cases favorable to the father can be taken with each of the 77,012 cases favorable to the son. That is, the number of cases favorable to both is $46{,}754 \times 77{,}012$. For a similar reason, the whole number of cases is $82{,}581 \times 98{,}650$. Therefore, the required probability is $\frac{46{,}754 \times 77{,}012}{82{,}581 \times 98{,}650}$.

The value of this fraction to five decimals places is readily obtained by logarithms, and is .44198.

Mortality Table.

The following table is taken from the *Actuaries' Table of Mortality*, prepared from data furnished by seventeen English Life Insurance Offices. It is based on the record of 62,537 assurances, and has been generally adopted by American Companies.

Age.	Number Living.	Number Dying.	Age.	Number Living.	Number Dying.	Age.	Number Living.	Number Dying.
10	100,000	676	40	78,653	815	70	35,837	2,327
11	99,324	674	41	77,838	826	71	33,510	2,351
12	98,650	672	42	77,012	839	72	31,159	2,362
13	97,978	671	43	76,173	857	73	28,797	2,358
14	97,307	671	44	75,316	881	74	26,439	2,339
15	96,636	671	45	74,435	909	75	24,100	2,303
16	95,965	672	46	73,526	944	76	21,797	2,249
17	95,293	673	47	72,582	981	77	19,548	2,179
18	94,620	675	48	71,601	1,021	78	17,369	2,092
19	93,945	677	49	70,580	1,063	79	15,277	1,987
20	93,268	680	50	69,517	1,108	80	13,290	1,866
21	92,588	683	51	68,409	1,156	81	11,424	1,730
22	91,905	686	52	67,253	1,207	82	9,694	1,582
23	91,219	690	53	66,046	1,261	83	8,112	1,427
24	90,529	694	54	64,785	1,316	84	6,685	1,268
25	89,835	698	55	63,469	1,375	85	5,417	1,111
26	89,137	703	56	62,094	1,436	86	4,306	958
27	88,434	708	57	60,658	1,497	87	3,348	811
28	87,726	714	58	59,161	1,561	88	2,537	673
29	87,012	720	59	57,600	1,627	89	1,864	545
30	86,292	727	60	55,973	1,698	90	1,319	427
31	85,565	734	61	54,275	1,770	91	892	322
32	84,831	742	62	52,505	1,844	92	570	231
33	84,089	750	63	50,661	1,917	93	339	155
34	83,339	758	64	48,744	1,990	94	184	95
35	82,581	767	65	46,754	2,061	95	89	52
36	81,814	776	66	44,693	2,128	96	37	24
37	81,038	785	67	42,565	2,191	97	13	9
38	80,253	795	68	40,374	2,246	98	4	3
39	79,458	805	69	38,128	2,291	99	1	1

EXERCISES.

1. With one die, what is the probability of throwing 6? Not 6? 6 three times in succession?

2. In a single throw with two dice, what is the probability of throwing an even number? At least 8? Not more than 5?

3. The letters *a*, *e*, *f*, *r*, are placed at random in a line. What is the probability that *fear* or *fare* will be written? That both vowels will come together?

4. If 52 cards be dealt to four players, what is the probability that a particular player will receive the four aces?

5. From a box containing 4 red balls, 6 black balls, and 7 white balls, 3 balls are drawn at random. What is the probability of drawing one ball of each color? 2 black and 1 white? 3 red?

6. If 6 coins be tossed, what is the probability that they will fall 4 heads and 2 tails? 3 heads and 3 tails?

7. Nine persons are seated at random at a round table. What is the probability that A and B will be seated together? That C will be seated between A and B?

8. If 4 different volumes of history, 3 of mathematics, and 6 of literature be placed at random on a shelf, what is the probability that all the volumes in the same subject will be placed together?

9. From a box containing tickets numbered 1, 2, 3, ⋯, 20, three tickets are drawn at random. What is the probability of drawing 2, 3, 5? 2, 3, and not 5? Neither 2, 3, nor 5? All even numbers? Consecutive numbers?

10–18. What are the odds in favor of the events whose probabilities are required in Exx. 1–9?

Referring to the accompanying table of mortality, find the probabilities of the events in Exx. 19–21:

19. That a man of 45 will live to be 50. To be 60. To be 70. To be 80. That he will die within 5 years. Within 10 years. Within 20 years.

20. That a man of 90 will live one year. Two years. Three years. Four years. Five years. At least five years.

21. At marriage, a man and his wife are 25 and 21, respectively. What is the probability that they will live to celebrate their silver wedding? Their golden wedding?

22. A representative of a firm sailed, first cabin, on a steamer which had a crew of 150 men, and which carried 150 first cabin and 250 second cabin passengers. On the voyage a man was lost. What is the probability, *to the firm,* that he was their representative? What, when a later report states that he was a passenger? What, when a still later report states that he was a first cabin passenger?

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.			
100	00 000	043	087	130	173	217	260	303	346	389		**44**	**43**	**42**
01	432	475	518	561	604	647	689	732	775	817	1	4.4	4.3	4.2
02	860	903	945	988	*030	*072	*115	*157	*199	*242	2	8.8	8.6	8.4
03	01 284	326	368	410	452	494	536	578	620	662	3	13.2	12.9	12.6
04	703	745	787	828	870	912	953	995	*036	*078	4	17.6	17.2	16.8
05	02 119	160	202	243	284	325	366	407	449	490	5	22.0	21.5	21.0
06	531	572	612	653	694	735	776	816	857	898	6	26.4	25.8	25.2
07	938	979	*019	*060	*100	*141	*181	*222	*262	*302	7	30.8	30.1	29.4
08	03 342	383	423	463	503	543	583	623	663	703	8	35.2	34.4	33.6
09	743	782	822	862	902	941	981	*021	*060	*100	9	39.6	38.7	37.8
110	04 139	179	218	258	297	336	376	415	454	493		**41**	**40**	**39**
11	532	571	610	650	689	727	766	805	844	883	1	4.1	4.0	3.9
12	922	961	999	*038	*077	*115	*154	*192	*231	*269	2	8.2	8.0	7.8
13	05 308	346	385	423	461	500	538	576	614	652	3	12.3	12.0	11.7
14	690	729	767	805	843	881	918	956	994	*032	4	16.4	16.0	15.6
15	06 070	108	145	183	221	258	296	333	371	408	5	20.5	20.0	19.5
16	446	483	521	558	595	633	670	707	744	781	6	24.6	24.0	23.4
17	819	856	893	930	967	*004	*041	*078	*115	*151	7	28.7	28.0	27.3
18	07 188	225	262	298	335	372	408	445	482	518	8	32.8	32.0	31.2
19	555	591	628	664	700	737	773	809	846	882	9	36.9	36.0	35.1
120	918	954	990	*027	*063	*099	*135	*171	*207	*243		**38**	**37**	**36**
21	08 279	314	350	386	422	458	493	529	565	600	1	3.8	3.7	3.6
22	636	672	707	743	778	814	849	884	920	955	2	7.6	7.4	7.2
23	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	3	11.4	11.1	10.8
24	09 342	377	412	447	482	517	552	587	621	656	4	15.2	14.8	14.4
25	691	726	760	795	830	864	899	934	968	*003	5	19.0	18.5	18.0
26	10 037	072	106	140	175	209	243	278	312	346	6	22.8	22.2	21.6
27	380	415	449	483	517	551	585	619	653	687	7	26.6	25.9	25.2
28	721	755	789	823	857	890	924	958	992	*025	8	30.4	29.6	28.8
29	11 059	093	126	160	193	227	261	294	327	361	9	34.2	33.3	32.4
130	394	428	461	494	528	561	594	628	661	694		**35**	**34**	**33**
31	727	760	793	826	860	893	926	959	992	*024	1	3.5	3.4	3.3
32	12 057	090	123	156	189	222	254	287	320	352	2	7.0	6.8	6.6
33	385	418	450	483	516	548	581	613	646	678	3	10.5	10.2	9.9
34	710	743	775	808	840	872	905	937	969	*001	4	14.0	13.6	13.2
35	13 033	066	098	130	162	194	226	258	290	322	5	17.5	17.0	16.5
36	354	386	418	450	481	513	545	577	609	640	6	21.0	20.4	19.8
37	672	704	735	767	799	830	862	893	925	956	7	24.5	23.8	23.1
38	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	8	28.0	27.2	26.4
39	14 301	333	364	395	426	457	489	520	551	582	9	31.5	30.6	29.7
140	613	644	675	706	737	768	799	829	860	891		**32**	**31**	**30**
41	922	953	983	*014	*045	*076	*106	*137	*168	*198	1	3.2	3.1	3.0
42	15 229	259	290	320	351	381	412	442	473	503	2	6.4	6.2	6.0
43	534	564	594	625	655	685	715	746	776	896	3	9.6	9.3	9.0
44	836	866	897	927	957	987	*017	*047	*077	*107	4	12.8	12.4	12.0
45	16 137	167	197	227	256	286	316	346	376	406	5	16.0	15.5	15.0
46	435	465	495	524	554	584	613	643	673	702	6	19.2	18.6	18.0
47	732	761	791	820	850	879	909	938	967	997	7	22.4	21.7	21.0
48	17 026	056	085	114	143	173	202	231	260	289	8	25.6	24.8	24.0
49	319	348	377	406	435	464	493	522	551	580	9	28.8	27.9	27.0
N.	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	**Pp. Pts.**			

N.	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
51	898	926	955	984	*013	*041	*070	*099	*127	*156
52	18 184	213	241	270	298	327	355	384	412	441
53	469	498	526	554	583	611	639	667	696	724
54	752	780	808	837	865	893	921	949	977	*005
55	19 033	061	089	117	145	173	201	229	257	285
56	312	340	368	396	424	451	479	507	535	562
57	590	618	645	673	700	728	756	783	811	838
58	866	893	921	948	976	*003	*030	*058	*085	*112
59	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
61	683	710	737	763	790	817	844	871	898	925
62	952	978	*005	*032	*059	*085	*112	*139	*165	*192
63	21 219	245	272	299	325	352	378	405	431	458
64	484	511	537	564	590	617	643	669	696	722
65	748	775	801	827	854	880	906	932	958	985
66	22 011	037	063	089	115	141	167	194	220	246
67	272	298	324	350	376	401	427	453	479	505
68	531	557	583	608	634	660	686	712	737	763
69	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
71	300	325	350	376	401	426	452	477	502	528
72	553	578	603	629	654	679	704	729	754	779
73	805	830	855	880	905	930	955	980	*005	*030
74	24 055	080	105	130	155	180	204	229	254	279
75	304	329	353	378	403	428	452	477	502	527
76	551	576	601	625	650	674	699	724	748	773
77	797	822	846	871	895	920	944	969	993	*018
78	25 042	066	091	115	139	164	188	212	237	261
79	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
81	768	792	816	840	864	888	912	935	959	983
82	26 007	031	055	079	102	126	150	174	198	221
83	245	269	293	316	340	364	387	411	435	458
84	482	505	529	553	576	600	623	647	670	694
85	717	741	764	788	811	834	858	881	905	928
86	951	975	998	*021	*045	*068	*091	*114	*138	*161
87	27 184	207	231	254	277	300	323	346	370	393
88	416	439	462	485	508	531	554	577	600	623
89	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
91	28 103	126	149	171	194	217	240	262	285	307
92	330	353	375	398	421	443	466	488	511	533
93	556	578	601	623	646	668	691	713	735	758
94	780	803	825	847	870	892	914	937	959	981
95	29 003	026	048	070	092	115	137	159	181	203
96	226	248	270	292	314	336	358	380	403	425
97	447	469	491	513	535	557	579	601	623	645
98	667	688	710	732	754	776	798	820	842	863
99	885	907	929	951	973	994	*016	*038	*060	*081

Pp. Pts.

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

N. 0 1 2 3 4 5 6 7 8 9 Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
01	320	341	363	384	406	428	449	471	492	514
02	535	557	578	600	621	643	664	685	707	728
03	750	771	792	814	835	856	878	899	920	942
04	963	984	*006	*027	*048	*069	*091	*112	*133	*154
05	31 175	197	218	239	260	281	302	323	345	366
06	387	408	429	450	471	492	513	534	555	576
07	597	618	639	660	681	702	723	744	765	785
08	806	827	848	869	890	911	931	952	973	994
09	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
11	428	449	469	490	510	531	552	572	593	613
12	634	654	675	695	715	736	756	777	797	818
13	838	858	879	899	919	940	960	980	*001	*021
14	33 041	062	082	102	122	143	163	183	203	224
15	244	264	284	304	325	345	365	385	405	425
16	445	465	486	506	526	546	566	586	606	626
17	646	666	686	706	726	746	766	786	806	826
18	846	866	885	905	925	945	965	985	*005	*025
19	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
21	439	459	479	498	518	537	557	577	596	616
22	635	655	674	694	713	733	753	772	792	811
23	830	850	869	889	908	928	947	967	986	*005
24	35 025	044	064	083	102	122	141	160	180	199
25	218	238	257	276	295	315	334	353	372	392
26	411	430	449	468	488	507	526	545	564	583
27	603	622	641	660	679	698	717	736	755	774
28	793	813	832	851	870	889	908	927	946	965
29	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
31	361	380	399	418	436	455	474	493	511	530
32	549	568	586	605	624	642	661	680	698	717
33	736	754	773	791	810	829	847	866	884	903
34	922	940	959	977	996	*014	*033	*051	*070	*088
35	37 107	125	144	162	181	199	218	236	254	273
36	291	310	328	346	365	383	401	420	438	457
37	475	493	511	530	548	566	585	603	621	639
38	658	676	694	712	731	749	767	785	803	822
39	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
41	202	220	238	256	274	292	310	328	346	364
42	382	399	417	435	453	471	489	507	525	543
43	561	578	596	614	632	650	668	686	703	721
44	739	757	775	792	810	828	846	863	881	899
45	917	934	952	970	987	*005	*023	*041	*058	*076
46	39 094	111	129	146	164	182	199	217	235	252
47	270	287	305	322	340	358	375	393	410	428
48	445	463	480	498	515	533	550	568	585	602
49	620	637	655	672	690	707	724	742	759	777
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

	22	2	20	19	18	17
1	2.2	2.	2.0	1.9	1.8	1.7
2	4.4	4.	4.0	3.8	3.6	3.4
3	6.6	6.	6.0	5.7	5.4	5.1
4	8.8	8.	8.0	7.6	7.2	6.8
5	11.0	10.	10.0	9.5	9.0	8.5
6	13.2	12.	12.0	11.4	10.8	10.2
7	15.4	14.	14.0	13.3	12.6	11.9
8	17.6	16.	16.0	15.2	14.4	13.6
9	19.8	18.	18.0	17.1	16.2	15.3

250-299

N.	0	1	2	3	4	5	6	7	8	9	Pp.	Pts.
250	39 794	811	829	846	863	881	898	915	933	950		18
51	967	985	*002	*019	*037	*054	*071	*088	*106	*123	1	1.8
52	40 140	157	175	192	209	226	243	261	278	295	2	3.6
53	312	329	346	364	381	398	415	432	449	466	3	5.4
54	483	500	518	535	552	569	586	603	620	637	4	7.2
55	654	671	688	705	722	739	756	773	790	807	5	9.0
56	824	841	858	875	892	909	926	943	960	976	6	10.8
57	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	7	12.6
58	41 162	179	196	212	229	246	263	280	296	313	8	14.4
59	330	347	363	380	397	414	430	447	464	481	9	16.2
260	497	514	531	547	564	581	597	614	631	647		17
61	664	681	697	714	731	747	764	780	797	814	1	1.7
62	830	847	863	880	896	913	929	946	963	979	2	3.4
63	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	3	5.1
64	42 160	177	193	210	226	243	259	275	292	308	4	6.8
65	325	341	357	374	390	406	423	439	455	472	5	8.5
66	488	504	521	537	553	570	586	602	619	635	6	10.2
67	651	667	684	700	716	732	749	765	781	797	7	11.9
68	813	830	846	862	878	894	911	927	943	959	8	13.6
69	975	991	*008	*024	*040	*056	*072	*088	*104	*120	9	15.3
270	43 136	152	169	185	201	217	233	249	265	281		16
71	297	313	329	345	361	377	393	409	425	441	1	1.6
72	457	473	489	505	521	537	553	569	584	600	2	3.2
73	616	632	648	664	680	696	712	727	743	759	3	4.8
74	775	791	807	823	838	854	870	886	902	917	4	6.4
75	933	949	965	981	996	*012	*028	*044	*059	*075	5	8.0
76	44 091	107	122	138	154	170	185	201	217	232	6	9.6
77	248	264	279	295	311	326	342	358	373	389	7	11.2
78	404	420	436	451	467	483	498	514	529	545	8	12.8
79	560	576	592	607	623	638	654	669	685	700	9	14.4
280	716	731	747	762	778	793	809	824	840	855		15
81	871	886	902	917	932	948	963	979	994	*010	1	1.5
82	45 025	040	056	071	086	102	117	133	148	163	2	3.0
83	179	194	209	225	240	255	271	286	301	317	3	4.5
84	332	347	362	378	393	408	423	439	454	469	4	6.0
85	484	500	515	530	545	561	576	591	606	621	5	7.5
86	637	652	667	682	697	712	728	743	758	773	6	9.0
87	788	803	818	834	849	864	879	894	909	924	7	10.5
88	939	954	969	984	*000	*015	*030	*045	*060	*075	8	12.0
89	46 090	105	120	135	150	165	180	195	210	225	9	13.5
290	240	255	270	285	300	315	330	345	359	374		14
91	389	404	419	434	449	464	479	494	509	523	1	1.4
92	538	553	568	583	598	613	627	642	657	672	2	2.8
93	687	702	716	731	746	761	776	790	805	820	3	4.2
94	835	850	864	879	894	909	923	938	953	967	4	5.6
95	982	997	*012	*026	*041	*056	*070	*085	*100	*114	5	7.0
96	47 129	144	159	173	188	202	217	232	246	261	6	8.4
97	276	290	305	319	334	349	363	378	392	407	7	9.8
98	422	436	451	465	480	494	509	524	538	553	8	11.2
99	*567*	*582*	596	611	625	640	654	669	683	698	9	12.6
N.	*0*	1	2	3	4	5	6	7	8	9	Pp.Pts.	

N.	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
01	857	871	885	900	914	929	943	958	972	986
02	48 001	015	029	044	058	073	087	101	116	130
03	144	159	173	187	202	216	230	244	259	273
04	287	302	316	330	344	359	373	387	401	416
05	430	444	458	473	487	501	515	530	544	558
06	572	586	601	615	629	643	657	671	686	700
07	714	728	742	756	770	785	799	813	827	841
08	855	869	883	897	911	926	940	954	968	982
09	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
11	276	290	304	318	332	346	360	374	388	402
12	415	429	443	457	471	485	499	513	527	541
13	554	568	582	596	610	624	638	651	665	679
14	693	707	721	734	748	762	776	790	803	817
15	831	845	859	872	886	900	914	927	941	955
16	969	982	996	*010	*024	*037	*051	*065	*079	*092
17	50 106	120	133	147	161	174	188	202	215	229
18	243	256	270	284	297	311	325	338	352	365
19	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
21	651	664	678	691	705	718	732	745	759	772
22	786	799	813	826	840	853	866	880	893	907
23	920	934	947	961	974	987	*001	*014	*028	*041
24	51 055	068	081	095	108	121	135	148	162	175
25	188	202	215	228	242	255	268	282	295	308
26	322	335	348	362	375	388	402	415	428	441
27	455	468	481	495	508	521	534	548	561	574
28	587	601	614	627	640	654	667	680	693	706
29	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
31	983	996	*009	*022	*035	*048	*061	*075	*088	*101
32	52 114	127	140	153	166	179	192	205	218	231
33	244	257	270	284	297	310	323	336	349	362
34	375	388	401	414	427	440	453	466	479	492
35	504	517	530	543	556	569	582	595	608	621
36	634	647	660	673	686	699	711	724	737	750
37	763	776	789	802	815	827	840	853	866	879
38	892	905	917	930	943	956	969	982	994	*007
39	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
41	275	288	301	314	326	339	352	364	377	390
42	403	415	428	441	453	466	479	491	504	517
43	529	542	555	567	580	593	605	618	631	643
44	656	668	681	694	706	719	732	744	757	769
45	782	794	807	820	832	845	857	870	882	895
46	908	920	933	945	958	970	983	995	*008	*020
47	54 033	045	058	070	083	095	108	120	133	145
48	158	170	183	195	208	220	233	245	258	270
49	283	295	307	320	332	345	357	370	382	394
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

	15
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2	3.0
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4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
450	65 321	331	341	350	360	369	379	389	398	408	
51	418	427	437	447	456	466	475	485	495	504	
52	514	523	533	543	552	562	571	581	591	600	
53	610	619	629	639	648	658	667	677	686	696	
54	706	715	725	734	744	753	763	772	782	792	
55	801	811	820	830	839	849	858	868	877	887	10
56	896	906	916	925	935	944	954	963	973	982	1 1.0
57	992	*001	*011	*020	*030	*039	*049	*058	*068	*077	2 2.0
58	66 087	096	106	115	124	134	143	153	162	172	3 3.0
59	181	191	200	210	219	229	238	247	257	266	4 4.0
460	276	285	295	304	314	323	332	342	351	361	5 5.0
61	370	380	389	398	408	417	427	436	445	455	6 6.0
62	464	474	483	492	502	511	521	530	539	549	7 7.0
63	558	567	577	586	596	605	614	624	633	642	8 8.0
64	652	661	671	680	689	699	708	717	727	736	9 9.0
65	745	755	764	773	783	792	801	811	820	829	
66	839	848	857	867	876	885	894	904	913	922	
67	932	941	950	960	969	978	987	997	*006	*015	
68	67 025	034	043	052	062	071	080	089	099	108	
69	117	127	136	145	154	164	173	182	191	201	
470	210	219	228	237	247	256	265	274	284	293	9
71	302	311	321	330	339	348	357	367	376	385	1 0.9
72	394	403	413	422	431	440	449	459	468	477	2 1.8
73	486	495	504	514	523	532	541	550	560	569	3 2.7
74	578	587	596	605	614	624	633	642	651	660	4 3.6
75	669	679	688	697	706	715	724	733	742	752	5 4.5
76	761	770	779	788	797	806	815	825	834	843	6 5.4
77	852	861	870	879	888	897	906	916	925	934	7 6.3
78	943	952	961	970	979	988	997	*006	*015	*024	8 7.2
79	68 034	043	052	061	070	079	088	097	106	115	9 8.1
480	124	133	142	151	160	169	178	187	196	205	
81	215	224	233	242	251	260	269	278	287	296	
82	305	314	323	332	341	350	359	368	377	386	
83	395	404	413	422	431	440	449	458	467	476	
84	485	494	502	511	520	529	538	547	556	565	
85	574	583	592	601	610	619	628	637	646	655	
86	664	673	681	690	699	708	717	726	735	744	8
87	753	762	771	780	789	797	806	815	824	833	1 0.8
88	842	851	860	869	878	886	895	904	913	922	2 1.6
89	931	940	949	958	966	975	984	993	*002	*011	3 2.4
490	69 020	028	037	046	055	064	073	082	090	099	4 3.2
91	108	117	126	135	144	152	161	170	179	188	5 4.0
92	197	205	214	223	232	241	249	258	267	276	6 4.8
93	285	294	302	311	320	329	338	346	355	364	7 5.6
94	373	381	390	399	408	417	425	434	443	452	8 6.4
95	461	469	478	487	496	504	513	522	531	539	9 7.2
96	548	557	566	574	583	592	601	609	618	627	
97	636	644	653	662	671	679	688	697	705	714	
98	723	732	740	749	758	767	775	784	793	801	
99	810	819	827	836	845	854	862	871	880	888	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
500	69 897	906	914	923	932	940	949	958	966	975	
01	984	992	*001	*010	*018	*027	*036	*044	*053	*062	
02	70 070	079	088	096	105̅	114	122	131	140	148	
03	157	165	174	183	191	200	209	217	226	234	
04	243	252	260	269	278	286	295̅	303	312	321	
05	329	338	346	355̅	364	372	381	389	398	406	9
06	415	424	432	441	449	458	467	475	484	492	1 0.9
07	501	509	518	526	535	544	552	561	569	578	2 1.8
08	586	595̅	603	612	621	629	638	646	655̅	663	3 2.7
09	672	680	689	697	706	714	723	731	740	749	4 3.6
510	757	766	774	783	791	800	808	817	825	834	5 4.5
11	842	851	859	868	876	885̅	893	902	910	919	6 5.4
12	927	935	944	952	961	969	978	986	995̅	*003	7 6.3
13	71 012	020	029	037	046	054	063	071	079	088	8 7.2
14	096	105̅	113	122	130	139	147	155	164	172	9 8.1
15	181	189	198	206	214	223	231	240	248	257	
16	265̅	273	282	290	299	307	315	324	332	341	
17	349	357	366	374	383	391	399	408	416	425̅	
18	433	441	45̅0	458	466	475̅	483	492	5̅00	508	
19	517	525	533	542	550	559	567	575	584	592	
520	600	609	617	625	634	642	650	659	667	675	8
21	684	692	700	709	717	725	734	742	750	759	1 0.8
22	767	775	784	792	800	809	817	825	834	842	2 1.6
23	850	858	867	875	883	892	900	908	917	925̅	3 2.4
24	933	941	95̅0	95̅8	966	975̅	983	991	999	*008	4 3.2
25	72 016	024	032	041	049	057	066	074	082	090	5 4.0
26	099	107	115	123	132	140	148	156	165̅	173	6 4.8
27	181	189	198	206	214	222	230	239	247	255	7 5.6
28	263	272	280	288	296	304	313	321	329	337	8 6.4
29	346	354	362	370	378	387	395̅	403	411	419	9 7.2
530	428	436	444	452	460	469	477	485̅	493	501	
31	509	518	526	534	542	550	558	567	575̅	583	
32	591	599	607	616	624	632	640	648	656	665̅	
33	673	681	689	697	705	713	722	730	738	746	
34	754	762	770	779	787	795̅	803	811	819	827	
35	835	843	852	860	868	876	884	892	900	908	
36	916	925̅	933	941	949	957	965	973	981	989	7
37	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	1 0.7
38	73 078	086	094	102	111	119	127	135̅	143	151	2 1.4
39	159	167	175̅	183	191	199	207	215	223	231	3 2.1
540	239	247	255	263	272	280	288	296	304	312	4 2.8
41	320	328	336	344	352	360	368	376	384	392	5 3.5
42	400	408	416	424	432	440	448	456	464	472	6 4.2
43	480	488	496	504	512	520	528	536	544	552	7 4.9
44	560	568	576	584	592	600	608	616	624	632	8 5.6
45	640	648	656	664	672	679	687	695	703	711	9 6.3
46	719	727	735	743	751	759	767	775̅	783	791	
47	799	807	815̅	823	830	838	846	854	862	870	
48	878	886	894	902	910	918	926	933	941	949	
49	957	965	973	981	989	997	*005̅	*013	*020	*028	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
51	115	123	131	139	147	155	162	170	178	186
52	194	202	210	218	225	233	241	249	257	265
53	273	280	288	296	304	312	320	327	335	343
54	351	359	367	374	382	390	398	406	414	421
55	429	437	445	453	461	468	476	484	492	500
56	507	515	523	531	539	547	554	562	570	578
57	586	593	601	609	617	624	632	640	648	656
58	663	671	679	687	695	702	710	718	726	733
59	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
61	896	904	912	920	927	935	943	950	958	966
62	974	981	989	997	*005	*012	*020	*028	*035	*043
63	75 051	059	066	074	082	089	097	105	113	120
64	128	136	143	151	159	166	174	182	189	197
65	205	213	220	228	236	243	251	259	266	274
66	282	289	297	305	312	320	328	335	343	351
67	358	366	374	381	389	397	404	412	420	427
68	435	442	450	458	465	473	481	488	496	504
69	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
71	664	671	679	686	694	702	709	717	724	732
72	740	747	755	762	770	778	785	793	800	808
73	815	823	831	838	846	853	861	868	876	884
74	891	899	906	914	921	929	937	944	952	959
75	967	974	982	989	997	*005	*012	*020	*027	*035
76	76 042	050	057	065	072	080	087	095	103	110
77	118	125	133	140	148	155	163	170	178	185
78	193	200	208	215	223	230	238	245	253	260
79	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
81	418	425	433	440	448	455	462	470	477	485
82	492	500	507	515	522	530	537	545	552	559
83	567	574	582	589	597	604	612	619	626	634
84	641	649	656	664	671	678	686	693	701	708
85	716	723	730	738	745	753	760	768	775	782
86	790	797	805	812	819	827	834	842	849	856
87	864	871	879	886	893	901	908	916	923	930
88	938	945	953	960	967	975	982	989	997	*004
89	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
91	159	166	173	181	188	195	203	210	217	225
92	232	240	247	254	262	269	276	283	291	298
93	305	313	320	327	335	342	349	357	364	371
94	379	386	393	401	408	415	422	430	437	444
95	452	459	466	474	481	488	495	503	510	517
96	525	532	539	546	554	561	568	576	583	590
97	597	605	612	619	627	634	641	648	656	663
98	670	677	685	692	699	706	714	721	728	735
99	743	750	757	764	772	779	786	793	801	808
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
600	77 815	822	830	837	844	851	859	866	873	880	
01	887	895	902	909	916	924	931	938	945	952	
02	960	967	974	981	988	996	*003	*010	*017	*025	
03	78 032	039	046	053	061	068	075	082	089	097	
04	104	111	118	125	132	140	147	154	161	168	
05	176	183	190	197	204	211	219	226	233	240	8
06	247	254	262	269	276	283	290	297	305	312	1 0.8
07	319	326	333	340	347	355	362	369	376	383	2 1.6
08	390	398	405	412	419	426	433	440	447	455	3 2.4
09	462	469	476	483	490	497	504	512	519	526	4 3.2
610	533	540	547	554	561	569	576	583	590	597	5 4.0
11	604	611	618	625	633	640	647	654	661	668	6 4.8
12	675	682	689	696	704	711	718	725	732	739	7 5.6
13	746	753	760	767	774	781	789	796	803	810	8 6.4
14	817	824	831	838	845	852	859	866	873	880	9 7.2
15	888	895	902	909	916	923	930	937	944	951	
16	958	965	972	979	986	993	*000	*007	*014	*021	
17	79 029	036	043	050	057	064	071	078	085	092	
18	099	106	113	120	127	134	141	148	155	162	
19	169	176	183	190	197	204	211	218	225	232	
620	239	246	253	260	267	274	281	288	295	302	7
21	309	316	323	330	337	344	351	358	365	372	1 0.7
22	379	386	393	400	407	414	421	428	435	442	2 1.4
23	449	456	463	470	477	484	491	498	505	511	3 2.1
24	518	525	532	539	546	553	560	567	574	581	4 2.8
25	588	595	602	609	616	623	630	637	644	650	5 3.5
26	657	664	671	678	685	692	699	706	713	720	6 4.2
27	727	734	741	748	754	761	768	775	782	789	7 4.9
28	796	803	810	817	824	831	837	844	851	858	8 5.6
29	865	872	879	886	893	900	906	913	920	927	9 6.3
630	934	941	948	955	962	969	975	982	989	996	
31	80 003	010	017	024	030	037	044	051	058	065	
32	072	079	085	092	099	106	113	120	127	134	
33	140	147	154	161	168	175	182	188	195	202	
34	209	216	223	229	236	243	250	257	264	271	
35	277	284	291	298	305	312	318	325	332	339	
36	346	353	359	366	373	380	387	393	400	407	6
37	414	421	428	434	441	448	455	462	468	475	1 0.6
38	482	489	496	502	509	516	523	530	536	543	2 1.2
39	550	557	564	570	577	584	591	598	604	611	3 1.8
640	618	625	632	638	645	652	659	665	672	679	4 2.4
41	686	693	699	706	713	720	726	733	740	747	5 3.0
42	754	760	767	774	781	787	794	801	808	814	6 3.6
43	821	828	835	841	848	855	862	868	875	882	7 4.2
44	889	895	902	909	916	922	929	936	943	949	8 4.8
45	956	963	969	976	983	990	996	*003	*010	*017	9 5.4
46	81 023	030	037	043	050	057	064	070	077	084	
47	090	097	104	111	117	124	131	137	144	151	
48	158	164	171	178	184	191	198	204	211	218	
49	224	231	238	245	251	258	265	271	278	285	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
650	81 291	298	305	311	318	325	331	338	345	351	
51	358	365	371	378	385	391	398	405	411	418	
52	425	431	438	445	451	458	465	471	478	485	
53	491	498	505	511	518	525	531	538	544	551	
54	558	564	571	578	584	591	598	604	611	617	
55	624	631	637	644	651	657	664	671	677	684	
56	690	697	704	710	717	723	730	737	743	750	
57	757	763	770	776	783	790	796	803	809	816	
58	823	829	836	842	849	856	862	869	875	882	
59	889	895	902	908	915	921	928	935	941	948	
660	954	961	968	974	981	987	994	*000	*007	*014	7
61	82 020	027	033	040	046	053	060	066	073	079	1 0.7
62	086	092	099	105	112	119	125	132	138	145	2 1.4
63	151	158	164	171	178	184	191	197	204	210	3 2.1
64	217	223	230	236	243	249	256	263	269	276	4 2.8
65	282	289	295	302	308	315	321	328	334	341	5 3.5
66	347	354	360	367	373	380	387	393	400	406	6 4.2
67	413	419	426	432	439	445	452	458	465	471	7 4.9
68	478	484	491	497	504	510	517	523	530	536	8 5.6
69	543	549	556	562	569	575	582	588	595	601	9 6.3
670	607	614	620	627	633	640	646	653	659	666	
71	672	679	685	692	698	705	711	718	724	730	
72	737	743	750	756	763	769	776	782	789	795	
73	802	808	814	821	827	834	840	847	853	860	
74	866	872	879	885	892	898	905	911	918	924	
75	930	937	943	950	956	963	969	975	982	988	
76	995	*001	*008	*014	*020	*027	*033	*040	*046	*052	
77	83 059	065	072	078	085	091	097	104	110	117	
78	123	129	136	142	149	155	161	168	174	181	
79	187	193	200	206	213	219	225	232	238	245	
680	251	257	264	270	276	283	289	296	302	308	
81	315	321	327	334	340	347	353	359	366	372	6
82	378	385	391	398	404	410	417	423	429	436	1 0.6
83	442	448	455	461	467	474	480	487	493	499	2 1.2
84	506	512	518	525	531	537	544	550	556	563	3 1.8
85	569	575	582	588	594	601	607	613	620	626	4 2.4
86	632	639	645	651	658	664	670	677	683	689	5 3.0
87	696	702	708	715	721	727	734	740	746	753	6 3.6
88	759	765	771	778	784	790	797	803	809	816	7 4.2
89	822	828	835	841	847	853	860	866	872	879	8 4.8
690	885	891	897	904	910	916	923	929	935	942	9 5.4
91	948	954	960	967	973	979	985	992	998	*004	
92	84 011	017	023	029	036	042	048	055	061	067	
93	073	080	086	092	098	105	111	117	123	130	
94	136	142	148	155	161	167	173	180	186	192	
95	198	205	211	217	223	230	236	242	248	255	
96	261	267	273	280	286	292	298	305	311	317	
97	323	330	336	342	348	354	361	367	373	379	
98	386	392	398	404	410	417	423	429	435	442	
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02	634	640	646	652	658	66$\bar{5}$	671	677	683	689
03	696	702	708	714	720	726	733	739	74$\bar{5}$	751
04	757	763	770	776	782	788	794	800	807	813
05	819	825	831	837	844	8$\bar{5}$0	856	862	868	874
06	880	887	893	899	905	911	917	924	930	936
07	942	948	954	960	967	973	979	98$\bar{5}$	991	997
08	85 003	009	016	022	028	034	040	046	052	058
09	06$\bar{5}$	071	077	083	089	095	101	107	114	120
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11	187	193	199	205	211	217	224	230	236	242
12	248	254	260	266	272	278	28$\bar{5}$	291	297	303
13	309	315	321	327	333	339	345	352	358	364
14	370	376	382	388	394	400	406	412	418	42$\bar{5}$
15	431	437	443	449	45$\bar{5}$	461	467	473	479	485
16	491	497	503	509	516	522	528	534	540	546
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18	612	618	62$\bar{5}$	631	637	643	649	65$\bar{5}$	661	667
19	673	679	68$\bar{5}$	691	697	703	709	715	721	727
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21	794	800	806	812	818	824	830	836	842	848
22	854	860	866	872	878	884	890	896	902	908
23	914	920	926	932	938	944	9$\bar{5}$0	956	962	968
24	974	980	986	992	998	*004	*010	*016	*022	*028
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27	153	159	165	171	177	183	189	195	201	207
28	213	219	225	231	237	243	249	25$\bar{5}$	261	267
29	273	279	28$\bar{5}$	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
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35	629	63$\bar{5}$	641	646	652	658	664	670	676	682
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37	747	753	759	764	770	776	782	788	794	800
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39	864	870	876	882	888	894	900	906	911	917
740	923	929	93$\bar{5}$	941	947	953	958	964	970	976
41	982	988	994	999	*005	*011	*017	*023	*029	*03$\bar{5}$
42	87 040	046	052	058	064	070	075	081	087	093
43	099	10$\bar{5}$	111	116	122	128	134	140	146	151
44	157	163	169	17$\bar{5}$	181	186	192	198	204	210
45	216	221	227	233	239	24$\bar{5}$	251	256	262	268
46	274	280	286	291	297	303	309	31$\bar{5}$	320	326
47	332	338	344	349	355	361	367	373	379	384
48	390	396	402	408	413	419	42$\bar{5}$	431	437	442
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52	622	628	633	639	645̄	651	656	662	668	674
53	679	685	691	697	703	708	714	720	726	731
54	737	743	749	754	760	766	772	777	783	789
55	795̄	800	806	812	818	823	829	835̄	841	846
56	852	858	864	869	875	881	887	892	898	904
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58	967	973	978	984	990	996	*001	*007	*013	*018
59	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
61	138	144	150̄	156	161	167	173	178	184	190
62	195	201	207	213	218	224	230	235	241	247
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64	309	315	321	326	332	338	343	349	355̄	360
65	366	372	377	383	389	395̄	400	406	412	417
66	423	429	434	440	446	451	457	463	468	474
67	480	485	491	497	502	508	513	519	525̄	530
68	536	542	547	553	559	564	570	576	581	587
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74	874	880	885	891	897	902	908	913	919	925̄
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77	89 042	048	053	059	064	070	076	081	087	092
78	098	104	109	115̄	120	126	131	137	143	148
79	154	159	165̄	170	176	182	187	193	198	204
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83	376	382	387	393	398	404	409	415̄	421	426
84	432	437	443	448	454	459	465̄	470	476	481
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86	542	548	553	559	564	570	575	581	586	592
87	597	603	609	614	620	625	631	636	642	647
88	653	658	664	669	675̄	680	686	691	697	702
89	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785̄	790	796	801	807	812
91	818	823	829	834	840	845	851	856	862	867
92	873	878	883	889	894	900	905	911	916	922
93	927	933	938	944	949	955̄	960	966	971	977
94	982	988	993	998	*004	*009	*015̄	*020	*026	*031
95	90 037	042	048	053	059	064	069	075̄	080	086
96	091	097	102	108	113	119	124	129	135̄	140
97	146	151	157	162	168	173	179	184	189	195̄
98	*200*	206	211	217	222	227	233	238	244	249
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02	417	423	428	434	439	445̄	450̄	455	461	466
03	472	477	482	488	493	499	504	509	515̄	520
04	526	531	536	542	547	553	558	563	569	574
05	580	585̄	590	596	601	607	612	617	623	628
06	634	639	644	650̄	655	660	666	671	677	682
07	687	693	698	703	709	714	720	725	730	736
08	741	747	752	757	763	768	773	779	784	789
09	795̄	800	806	811	816	822	827	832	838	843
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16	169	174	180	185̄	190	196	201	206	212	217
17	222	228	233	238	243	249	254	259	265̄	270
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820	381	387	392	397	403	408	413	418	424	429
21	434	440	445̄	450	455	461	466	471	477	482
22	487	492	498	503	508	514	519	524	529	535̄
23	540	545	551	556	561	566	572	577	582	587
24	593	598	603	609	614	619	624	630	635̄	640
25	645	651	656	661	666	672	677	682	687	693
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32	92 012	018	023	028	033	038	044	049	054	059
33	065̄	070	075̄	080	085	091	096	101	106	111
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41	480	485̄	490	495	500	505	511	516	521	526
42	531	536	542	547	552	557	562	567	572	578
43	583	588	593	598	603	609	614	619	624	629
44	634	639	645̄	650̄	655̄	660	665	670	675	681
45	686	691	696	701	706	711	716	722	727	732
46	737	742	747	752	758	763	768	773	778	783
47	788	793	799	804	809	814	819	824	829	834
48	840	845̄	850̄	855̄	860	865	870	875	881	886
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91	988	993	998	*002	*007	*012	*017	*022	*027	*032
92	95 036	041	046	051	056	061	066	071	075	080
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12	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
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14	095̄	099	104	109	114	118	123	128	133	137
15	142	147	152	156	161	166	171	175	180	185̄
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27	708	713	717	722	727	731	736	741	745	750
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35	081	086	090	095	100	104	109	114	118	123
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66	498	502	507	511	516	520	525	529	534	538	4 2.0
67	543	547	552	556	561	565	570	574	579	583	5 2.5
68	588	592	597	601	605	610	614	619	623	628	6 3.0
69	632	637	641	646	650	655	659	664	668	673	7 3.5
970	677	682	686	691	695	700	704	709	713	717	8 4.0
71	722	726	731	735	740	744	749	753	758	762	9 4.5
72	767	771	776	780	784	789	793	798	802	807	
73	811	816	820	825	829	834	838	843	847	851	
74	856	860	865	869	874	878	883	887	892	896	
75	900	905	909	914	918	923	927	932	936	941	
76	945	949	954	958	963	967	972	976	981	985	
77	989	994	998	*003	*007	*012	*016	*021	*025	*029	
78	99 034	038	043	047	052	056	061	065	069	074	
79	078	083	087	092	096	100	105	109	114	118	
980	123	127	131	136	140	145	149	154	158	162	
81	167	171	176	180	185	189	193	198	202	207	4
82	211	216	220	224	229	233	238	242	247	251	1 0.4
83	255	260	264	269	273	277	282	286	291	295	2 0.8
84	300	304	308	313	317	322	326	330	335	339	3 1.2
85	344	348	352	357	361	366	370	374	379	383	4 1.6
86	388	392	396	401	405	410	414	419	423	427	5 2.0
87	432	436	441	445	449	454	458	463	467	471	6 2.4
88	476	480	484	489	493	498	502	506	511	515	7 2.8
89	520	524	528	533	537	542	546	550	555	559	8 3.2
990	564	568	572	577	581	585	590	594	599	603	9 3.6
91	607	612	616	621	625	629	634	638	642	647	
92	651	656	660	664	669	673	677	682	686	691	
93	695	699	704	708	712	717	721	726	730	734	
94	739	743	747	752	756	760	765	769	774	778	
95	782	787	791	795	800	804	808	813	817	822	
96	826	830	835	839	843	848	852	856	861	865	
97	870	874	878	883	887	891	896	900	904	909	
98	913	917	922	926	930	935	939	944	948	952	
99	957	961	965	970	974	978	983	987	991	996	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

NAME	DATE	NAME	DATE